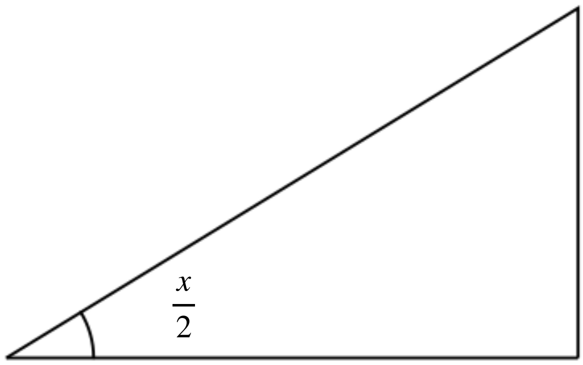
Tangent Half Angle Substitution

Described by the legendary Michael Spivak as “the world’s sneakiest substitution”.

First we need to consider some basic trigonometry to enable us to use this substitution.

Consider the right angled triangle.

Let $t = \tan\left(\frac{x}{2}\right)$



Derive expressions for the following:

$$\sin\left(\frac{x}{2}\right) =$$

$$\cos\left(\frac{x}{2}\right) =$$

Using the double angle formulae we note that

$\cos(x) =$

and

$\sin(x) =$

Finally, differentiating $t = \tan\left(\frac{x}{2}\right)$, we can obtain an expression for the differential $\mathrm{d}x$.

$$\frac{\mathrm{d}t}{\mathrm{d}x} =$$

$$\mathrm{d}x =$$

<u>Example:</u>

Find $\int \frac{1}{1+\sin(x)}\ \mathrm{d}x$

Example

I. Express in partial fractions $y = \dfrac{1}{2t^2 + 3t - 2}$

II. Hence, using the tangent half angle substitution find $\displaystyle\int \frac{1}{3\sin(x) - 4\cos(x)}\,\mathrm{d}x$

Exercises

1. Find $\displaystyle\int \frac{\cos(x)}{1+\cos(x)}\,\mathrm{d}x$

2. Find $\int \frac{2}{3\cos(x) + 4\sin(x)}\,\mathrm{d}x$

3. Consider the function $f(x) = \dfrac{1}{2 + 3\sin(x)}$

a) Show that on the interval $\left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$ $f(x)$ is monotonically decreasing.

b) Hence, or otherwise, deduce that $\dfrac{\pi}{12 + 9\sqrt{3}} \geq \displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2 + 3\sin(x)}\,\mathrm{dx} \geq \frac{\pi}{30}$

c) Find the exact value of $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2+3\sin(x)} \, \mathrm{dx}$ and verify that it does lie in the interval given above.