Modelling with Differential Equations

1) A large drop of castor oil spreads out on the surface of some still water. The resulting patch of oil is circular, and it is suggested that the rate of increase of the radius is proportional to time. At $t = 0$ seconds the radius is $r = 1$cm.
Derive a differential equation for the radius of the oil patch and then solve

2) An iron bar of length 1m has one end placed in a furnace. The temperature of the bar decreases with respect to the distance from that end and at a rate proportional to the distance. At $x = 10$ the temperature is decreasing at a rate of 15°C / cm. The temperature of the end placed in the furnace is 400°C.
Derive a differential equation for the temperature at a distance x along the bar and

3) The amount of painkiller in the body can be modelled by a differential equation. We assume:

- The drug is instantaneously absorbed by the body (on taking the painkiller).
- Renal clearance is the sole mechanic, for the removal of the pain killer.
- The rate of clearance is proportional to the quantity in the body.

With t in hours the mass of painkiller when $t = 0$ is 400mg. When the mass is 200mg the rate of clearance is 150mg/hr.
Derive a differential equation for the mass of painkiller at time t and then solve it.

4) A particle falls through a viscous liquid such that its deceleration is proportional to the square root of the velocity. The initial speed is 25ms^{-1} and when the speed is 16ms^{-1} the acceleration is -2 ms^{-2}.
Form a differential equation for the rate of change of velocity and solve it to find the velocity as a function of time.

5) A barrel of wine contains a tap 5cm above the base of the barrel.
As the tap is opened the rate of decrease of height is proportional to the square root of $(h - 5)$. The initial height of the wine in the barrel is 69cm and when the height is 60cm the height is decreasing at a rate of 2cms^{-1}.
Form a differential equation in terms of height and solve it.

6) A cup of tea is cooling at a rate proportional to the difference between the temperature of the tea and the ambient temperature.
Initially the room is at 20°C and the tea has a temperature of 80°C. The tea starts to cool at an initial rate of 0.5°C/m.
Form (and solve) a differential equation for the temperature of the tea.

7) Liquid is poured into a large circular cylinder of radius 2m at a rate of 20m^{3}s^{-1}. The liquid also leaks out of the base at a rate proportional to the height of the liquid in the cylinder.
Form a differential equation for this situation and find the general solution.

8) Bacteria is growing in a jelly culture of area 10cm^2. It is suggested that the rate of change of area of the bacteria is proportional to the product of the current area of bacteria and the remaining area of “free” food.
Form a differential equation for this situation and find its general solution.

$y(x) = \dfrac{10\mathrm{e}^{10kx}}{C + \mathrm{e}^{10kx}}$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{5}{\pi} - \dfrac{ky}{4\pi}$
$y(x) = C\mathrm{e}^{-\frac{kx}{4\pi}} + \dfrac{20}{k}$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = kx$
$y(x) = \dfrac{1}{16}(20 - x)^2$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky(10 - y)$
$y(x) = 400 - \dfrac{3}{4}x^2$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -k\sqrt{y - 5}$
$y(x) = 60\mathrm{e}^{-x/120} + 20$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -ky$
$y(x) = k\dfrac{x^2}{2} + 1$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -k\sqrt{y}$
$y(x) = 400\mathrm{e}^{-0.75x}$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -k(y - 20)$
$y(x) = \left(-\dfrac{x}{\sqrt{55}} + 8\right)^2 + 5$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -kx$

To make things challenging the differential equations to match all use x as the independent variable and y as the dependent variable.

Solutions

Situation	DE	Solution
1	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = kx$	$y(x) = k\dfrac{x^2}{2} + 1$
2	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -kx$	$y(x) = 400 - \dfrac{3}{4}x^2$
3	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -ky$	$y(x) = 400\mathrm{e}^{-0.75x}$
4	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -k\sqrt{y}$	$y(x) = \dfrac{1}{16}(20 - x)^2$
5	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -k\sqrt{y - 5}$	$y(x) = \left(-\dfrac{x}{\sqrt{55}} + 8\right)^2 + 5$
6	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -k(y - 20)$	$y(x) = 60\mathrm{e}^{-x/120} + 20$
7	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{5}{\pi} - \dfrac{ky}{4\pi}$	$y(x) = C\mathrm{e}^{-\frac{kx}{4\pi}} + \dfrac{20}{k}$
8	$\dfrac{\mathrm{d}y}{\mathrm{dx}} = ky(10 - y)$	$y(x) = \dfrac{10\mathrm{e}^{10kx}}{C + \mathrm{e}^{10kx}}$