<u>Matrix Transformations - Revision</u>

We can use matrices to transform one coordinate to another by matrix multiplication.

For example the matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the following effect on the coordinate (x, y):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

That is, the matrix T takes the coordinate (x, y) and transforms it to the coordinate $(ax + by, cx + dy)$. Put another way, the matrix T performs the transformation

$$x \mapsto ax + by$$
$$y \mapsto cx + dy$$

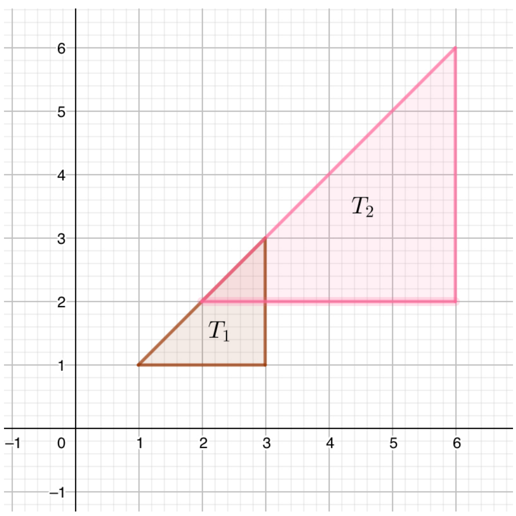
To see the effect of a transformation on a whole shape we can apply the matrix representing the transformation to each coordinate (expressed as a column vector).

<u>Example:</u>

For the triangle with coordinates $(1,1)$, $(3,1)$ and $(3,3)$ the enlargement by a scale factor of 2, centre the origin can be represented by the matrix $E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then,

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 2 & 2 & 6 \end{pmatrix}$$

giving a triangle of twice the size.
This is shown graphically below.

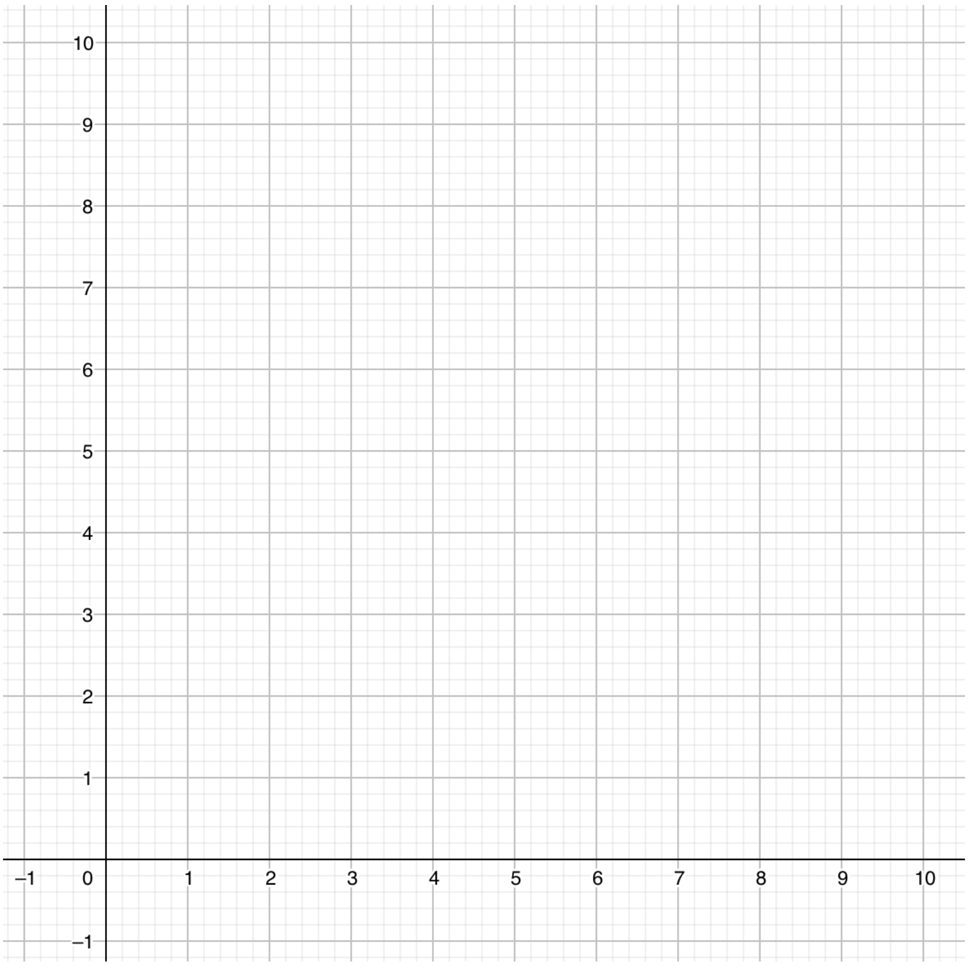


We say that T_1 is the "object" and T_2 is the "image" of T_1 under the transformation E.

Questions:

1) Write down the matrix which is equivalent to the mappings $x \mapsto 2x + y$ and $y \mapsto x + y$.

Show on the axes below the effect of this transformation on the rectangle worth coordinates $(1,1)$, $(3,1)$, $(3,4)$ and $(1,4)$. Show your matrix multiplication.



2) What is the image of the point $(3,5)$ under the transformation represented by the matrix $\begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}$.

<u>Standard Matrix Transformations to Know</u>

Identity transformation	$\begin{pmatrix} & \\ & \end{pmatrix}$
Enlargement, scale factor k, centre the origin	$\begin{pmatrix} & \\ & \end{pmatrix}$
Reflection in the line $x = 0$	$\begin{pmatrix} & \\ & \end{pmatrix}$
Reflection in the line $y = 0$	$\begin{pmatrix} & \\ & \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} & \\ & \end{pmatrix}$
Reflection in the line $y = -x$	$\begin{pmatrix} & \\ & \end{pmatrix}$
Rotation by $\theta°$ clockwise, centre the origin.	$\begin{pmatrix} & \\ & \end{pmatrix}$
Rotation by $\theta°$ anticlockwise, centre the origin.	$\begin{pmatrix} & \\ & \end{pmatrix}$

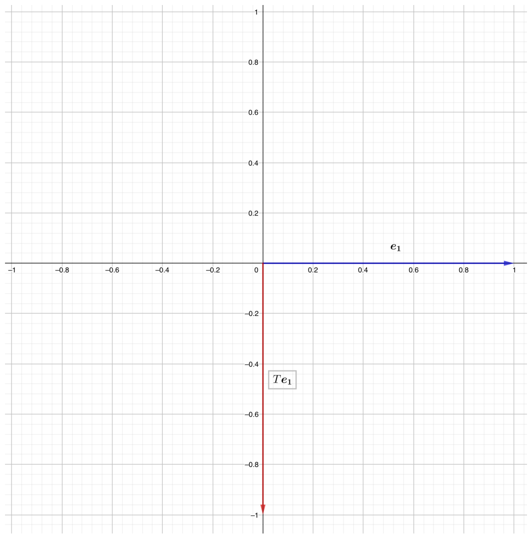
For the rotations you only need to be able to use the angles $90°$, $180°$ and $270°$.

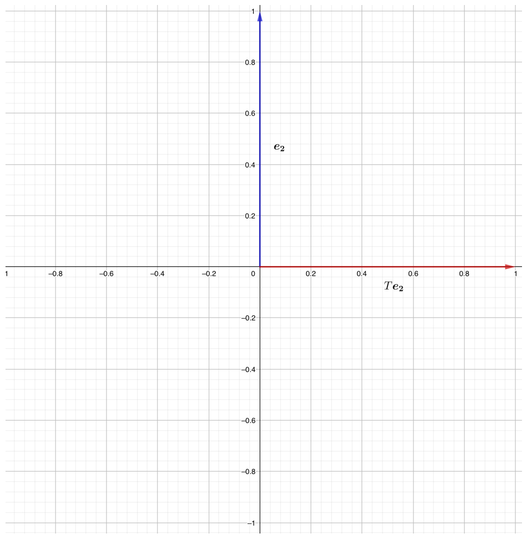
To determine a matrix for a given transformation we can examine the effect of the transformation on the unit vectors $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example:

The transformation T is a rotation $270°$ about the origin. To find the matrix representing this transformation we consider what happens to the unit vectors under this transformation.

The diagram on the left below shows $\mathbf{e_1}$ and $T\mathbf{e_1}$ (the vector from applying the transformation to $\mathbf{e_1}$





Hence, T maps $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}0\\-1\end{pmatrix}$.

Similarly, the diagram on the right above shows $\mathbf{e_2}$ and $T\mathbf{e_2}$, and so T maps $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}1\\0\end{pmatrix}$.

To obtain the transformation matrix we simply write the two image vectors side by side. So, the transformation T is represented by the matrix

$$T = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}.$$

Note: In questions concerning rotation, unless told otherwise rotations are performed anti-clockwise.

Example:

The matrix $\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$ is an enlargement, scale factor 4, centred the origin.

Note: When describing transformations given by a matrix give as much detail as you would in a GCSE transformations question.

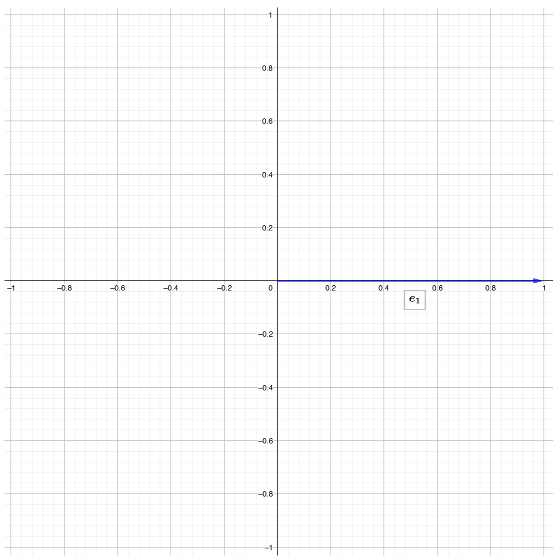
Questions:

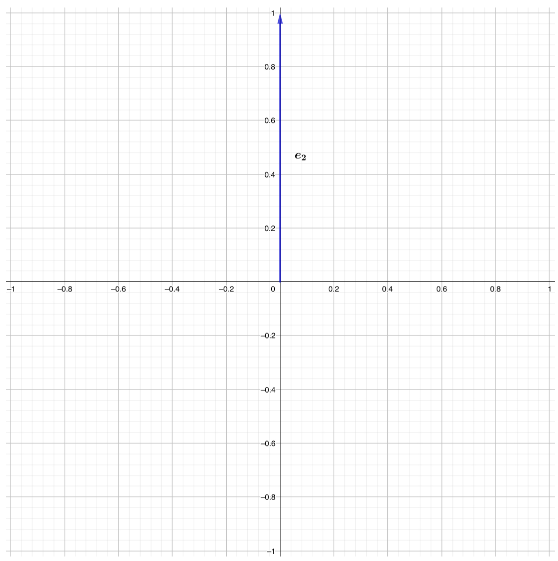
1)

a) Describe the transformation $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

b) Describe the transformation $N = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

2) Using the grids below find the matrix representing a reflection in the line $y = -x$





Combining Transformations

Suppose that we have a point L and we first apply one transformation T_1 to it, and then apply a second transformation T_2. This can be represented diagrammatically as below

$$L \xrightarrow{\;T_1\;} L' \xrightarrow{\;T_2\;} L''$$

The combined transformation would map L to L'' in one go.

To find the matrix transformation for the combined transformation, suppose the matrix A represents T_1 and the matrix B represents T_2, then the matrix representing the combined transformation is given by

$$BA$$

Example:

A transformation represented by the matrix $M = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ is followed by a transformation represented by the matrix $N = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Then the combined transformation is represented by

$$NM = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 9 & 6 \end{pmatrix}$$

Note: This is similar to the composition of functions. Suppose you are applying the transformations to a point L. Then applying M to L is equivalent to writing ML, you then apply N which is the same as calculating $N(ML) = NML$. Hence the matrix representing the combined transformation is NM.

Example:

A point $P(1,2)$ is transformed by $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, followed by a further transformation by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

The transformed transformation is then represented by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}$$

Hence, the image of P after both transformations is given by

$$\begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}.$$

Questions:

1) Find the matrix representing the transformation formed from a reflection in the $y-$ axis followed by a rotation by 90° anticlockwise.

2) Find the matrix representing the transformation formed from an enlargement, centre the origin, scale factor 2, followed by a rotation by 90° anticlockwise and then finally a reflection in the line $y = x$