# The Inverse of A $2 \times 2$ Matrix

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CHAPTER 1

# The Inverse of a General $2 \times 2$ Matrix

We want to find the inverse of the $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So this means finding the matrix $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ such that $AB = I = BA$. Of course, we want to find the entries $e, f, g$ and $h$ in terms of the entries of the matrix $A$. We have the following identity,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.1}$$

which leads to the following four simultaneous equations to be solved

$$ae + bg = 1 \tag{1.2}$$

$$af + bh = 0 \tag{1.3}$$

$$ce + dg = 0 \tag{1.4}$$

$$cf + dh = q \tag{1.5}$$

If we consider Equations (1.2) and (1.4) we can eliminate $g$ by multiplying (1.2) by $d$,

$$dae + dbg = d \tag{1.6}$$

and multiplying (1.4) by $b$ which gives

$$bce + bdg = 0 \tag{1.7}$$

Now subtracting, (1.6) − (1.7) leads to

$$dae - bce = d \tag{1.8}$$
$$\Rightarrow \quad e(da - bc) = d$$

which means that,

$$e = \frac{d}{da - bc} \tag{1.9}$$

Substituting (1.9) into (1.4) we have,

$$\frac{cd}{da - bc} + dg = 0,$$

meaning that

$$g = -\frac{c}{da - bc}. \tag{1.10}$$

So we now have two entries of the inverse matrix, it just remains to find the other two. Using (1.3) and (1.5) we can proceed in the same way and eliminate $f$.

$$(1.3) \times c \qquad caf + cbh = 0 \tag{1.11}$$
$$(1.5) \times a \qquad caf + adh = a. \tag{1.12}$$

Now, (1.12) − (1.11) results in,

$$adh - cbh = a$$
$$\Rightarrow \quad h(ad - cb) = a$$

giving that,

$$h = \frac{a}{ad - cb}. \tag{1.13}$$

Finally, substituting (1.13) into (1.3) we can obtain our final entry, since

$$af + \frac{ba}{ad - cb} = 0 \tag{1.14}$$

means that

$$f = -\frac{b}{ad - cb} \tag{1.15}$$

And so we have worked out the form of the inverse of a general 2 × 2 matrix, which combining (1.9), (1.10), (1.13) and (1.15) we now know to be

$$A^{-1} = B \tag{1.16}$$
$$= \begin{pmatrix} \frac{d}{da-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{da-bc} & \frac{a}{ad-cb} \end{pmatrix} \tag{1.17}$$
$$= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1.18}$$