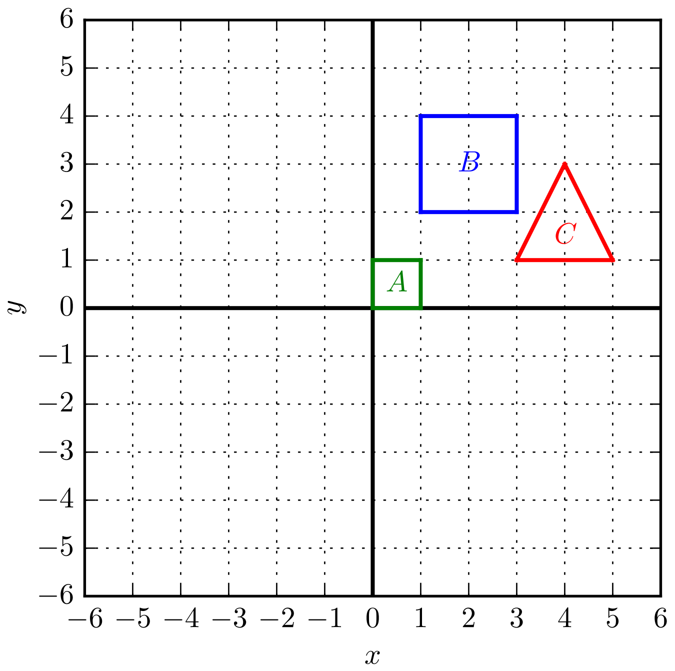
Introduction to Matrix Transformations

Matrices and their transformations underpin modern computer graphics. In this activity we explore the matrix representations for a variety of geometrical transformations.

Consider the shapes shown below.



We can describe these shapes using a matrix. Given the points $D(0,0)$, $E(1,0)$, $F(1,1)$ and $G(0,1)$ we can describe shape A by the matrix $S = (D\ E\ F\ G)$. That is, we define the 2×4 matrix from the coordinates of the vertices of A as below,

$$S_A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that the x-coordinates of the vertices are in the first row and the y-coordinates of the vertices are in the second row.

Write down similar matrices, S_B and S_C for the shapes B and C shown above.

$$S_B =$$

$$S_C =$$

We define the action of a matrix on a point by performing matrix multiplication. For example, consider the point $P(2,2)$ which, in column vector notation, is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and the matrix $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The action of U on P is calculated as follows

$$UP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Any geometric transformation can be represented as a matrix, it is this idea that we investigate in this activity.

<u>Investigation 1</u>

We wish to investigate the affect of the transformation T_1 which we define by the matrix

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using matrix multiplication we can investigate the effect of this transformation on the shape A. The columns of the product $M_1 S_A$ will be the coordinates of vertices of the transformed shape A'.

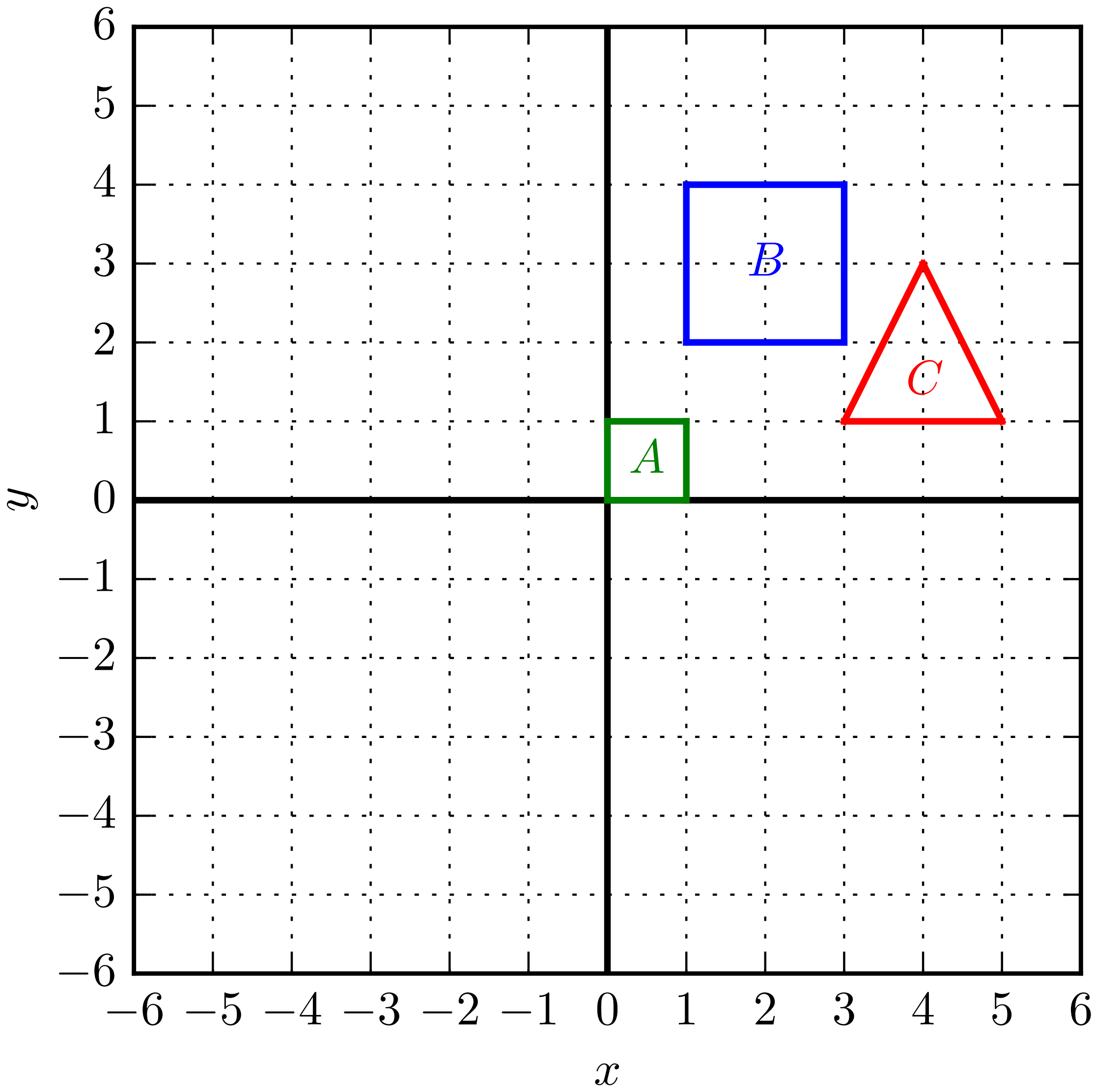
$$M_1 S_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

Now apply the transformation defined by the matrix M_1 to the matrices for shapes B and C.

$$M_1 S_B =$$

$$M_1 S_C =$$

Plot the transformed shapes A', B' and C' on the axes below.



What geometrical transformation does the matrix M_1 represent?

Investigation 2

We wish to investigate the affect of the transformation T_1 which we define by the matrix

$$M_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now apply the transformation defined by the matrix M_2 to the matrices for shapes A, B and C to generate the transformed shapes A', B' and C'.

$$M_2 S_A =$$

$$M_1 S_B =$$

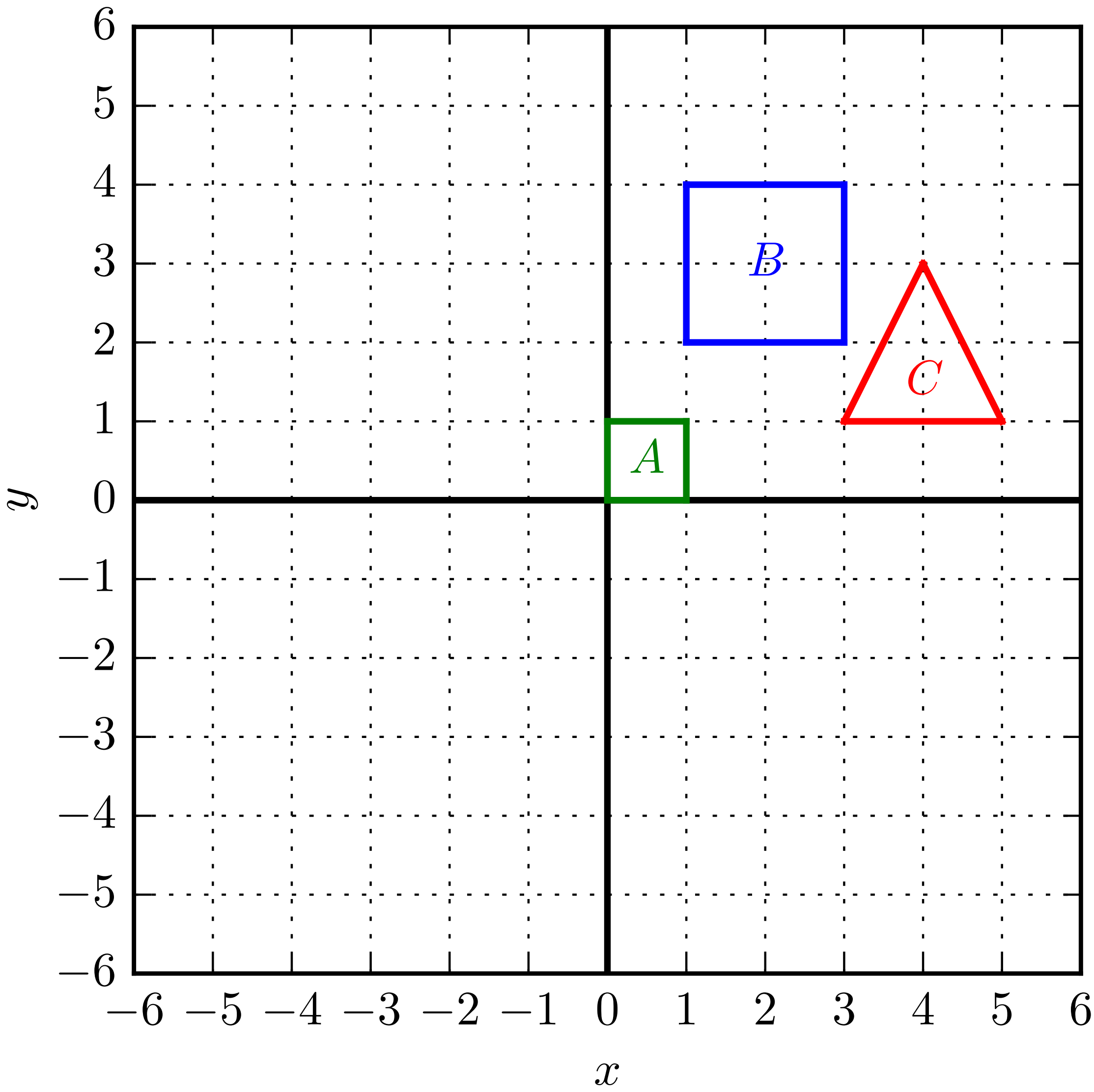
$$M_1 S_C =$$

Plot these transformed shapes on the grid below.

What geometric transformation does the matrix M_2 represent?

<u>Investigation 3</u>

We wish to investigate the affect of the transformation T_1 which we define by the matrix



$$M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now apply the transformation defined by the matrix M_3 to the matrices for shapes A, B and C to generate the transformed shapes A', B' and C'.

$$M_3 S_A =$$

$M_3 S_B =$

$M_3 S_C =$

Plot these transformed shapes on the grid below.

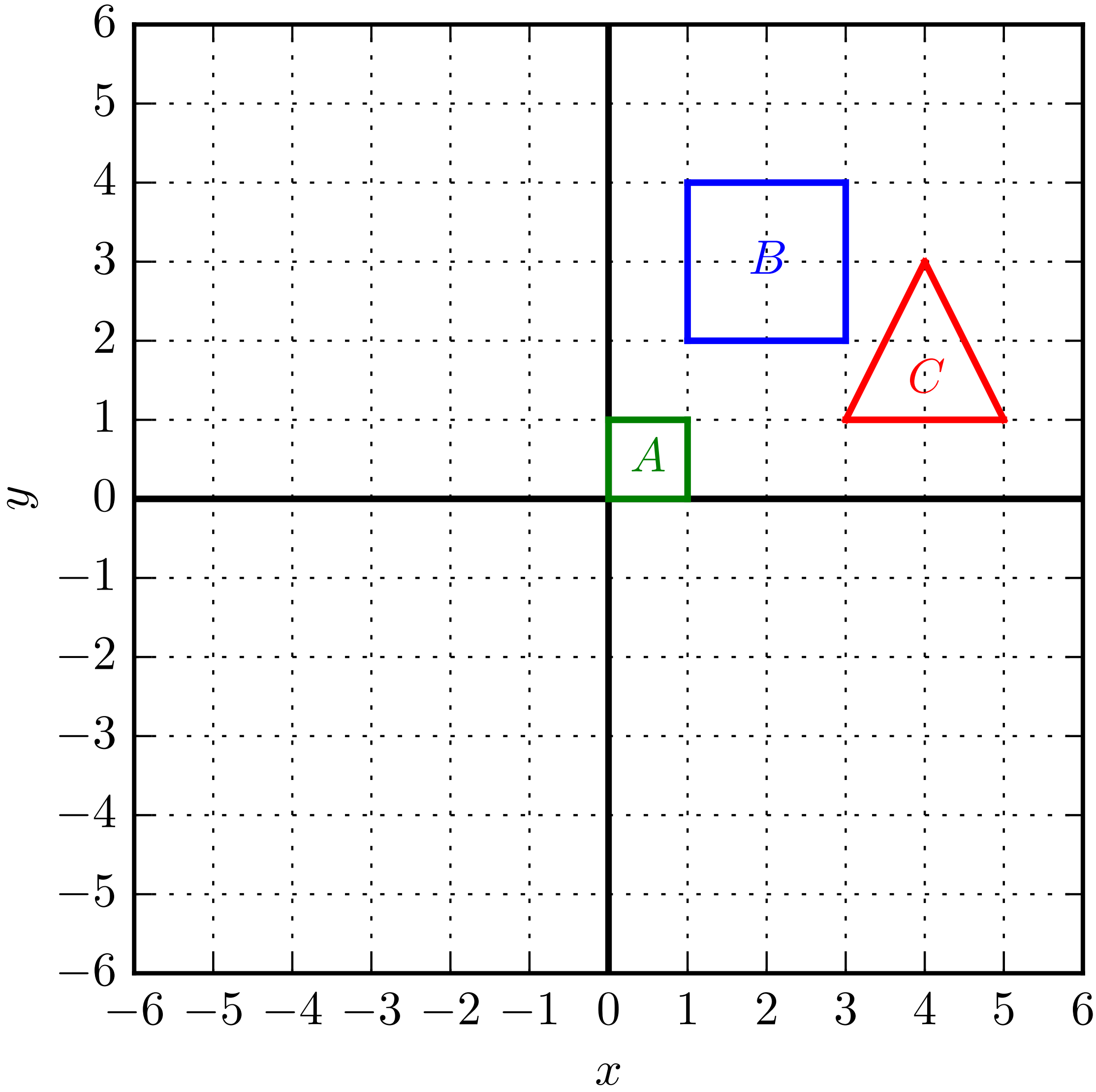
What geometric transformation does the matrix M_3 represent?

<u>Investigation 4</u>

We wish to investigate the affect of the transformation T_1 which we define by the matrix

$$M_4 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Now apply the transformation defined by the matrix M_4 to the matrices for shapes A, B and C to generate the transformed shapes A', B' and C'.

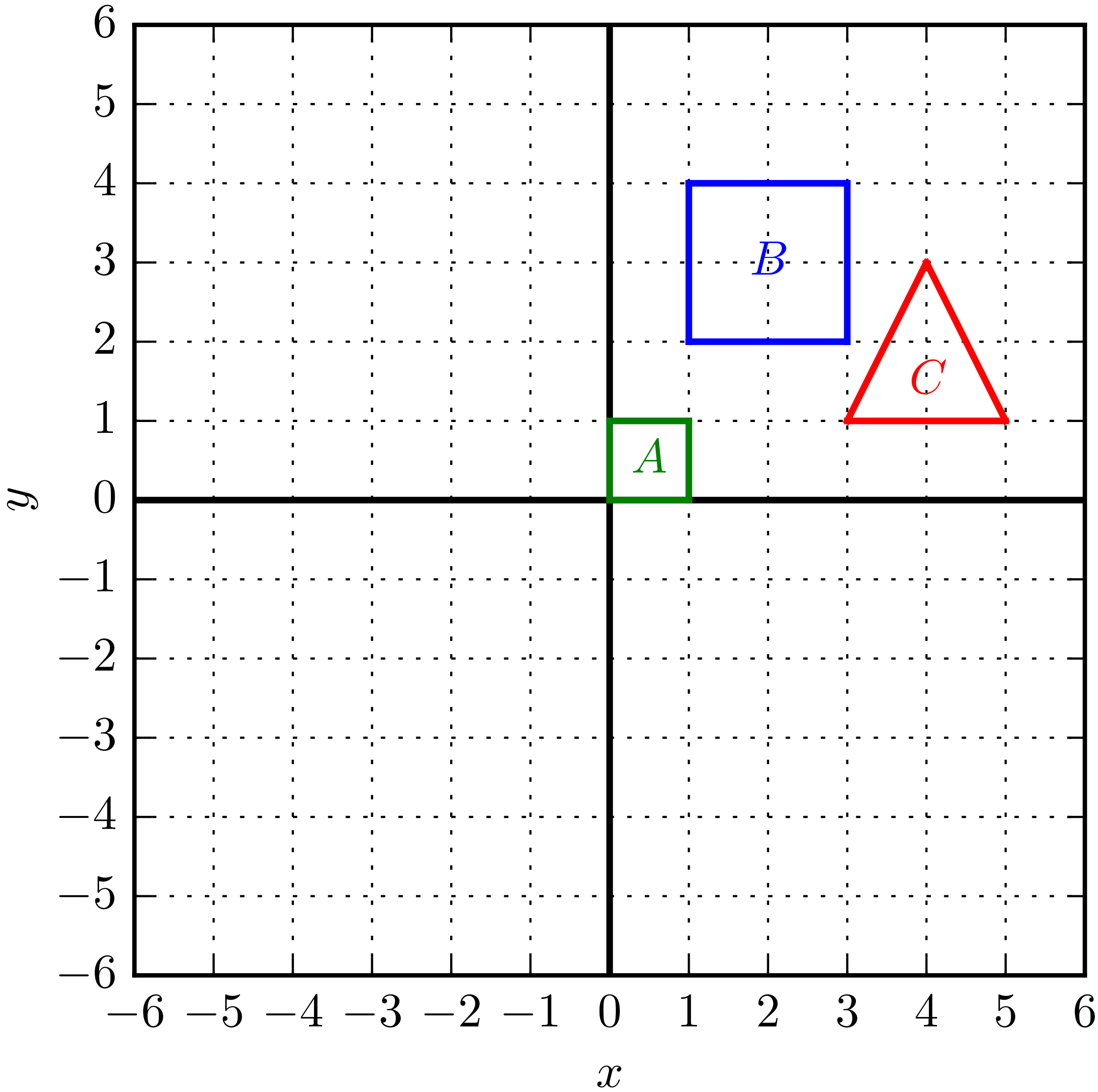


$M_4 S_A =$

$M_4 S_B =$

$M_4 S_C =$

Plot these transformed shapes on the grid below.



What geometric transformation does the matrix M_4 represent?

Investigation 5

We wish to investigate the affect of the transformation T_1 which we define by the matrix

$$M_5 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},$$

where $k \in \mathbb{R}$.

First consider the matrix $M_{(5,1)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Now apply the transformation defined by the matrix $M_{(5,1)}$ to the matrices for shapes A, B and C to generate the transformed shapes A', B' and C'.

$$M_{(5,1)}S_A =$$

$$M_{(5,1)}S_B =$$

$$M_{(5,1)}S_C =$$

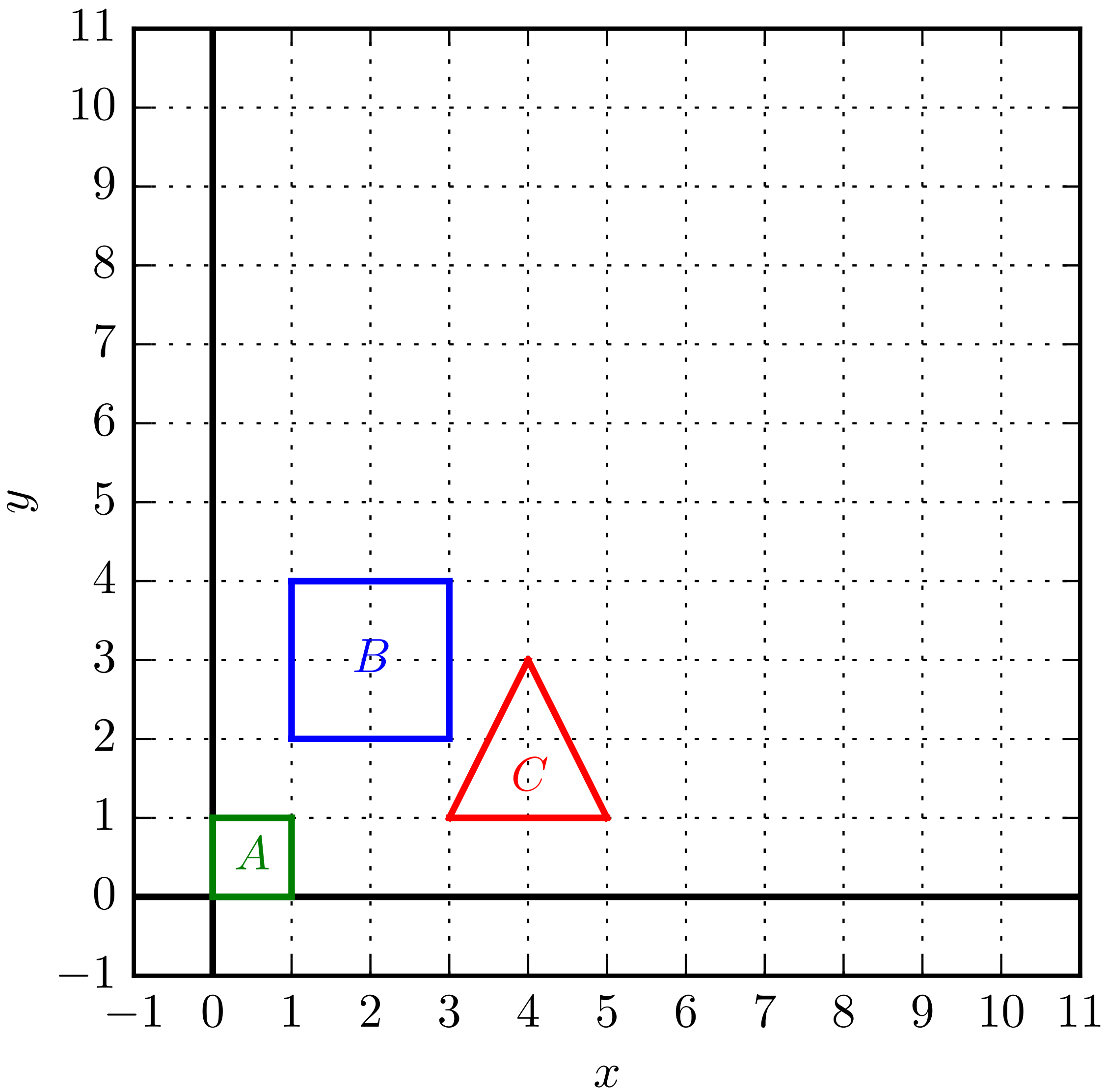
Now consider the matrix $M_{(5,2)} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Now apply the transformation defined by the matrix $M_{(5,2)}$ to the matrices for shapes A, B and C to generate the transformed shapes A'', B'' and C'.

$$M_{(5,2)}S_A =$$

$$M_{(5,2)}S_B =$$

$$M_{(5,2)}S_C =$$

Plot, and label, the shapes $A', B', C' A'', B''$ and C'' on the pair of axes on the next page.



Using the answers for the matrices $M_{(5,1)}$ and $M_{(5,2)}$ what geometric transformation does the general matrix,

$$M_5 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},$$

represent?

Investigation 6

We wish to investigate the affect of the transformation T_1 which we define by the matrix

$$M_6 = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix},$$

where $k \in \mathbb{R}$.

First consider the matrix $M_{(6,1)} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Now apply the transformation defined by the matrix $M_{(5,1)}$ to the matrices for shapes A, B and C to generate the transformed shapes A', B' and C'.

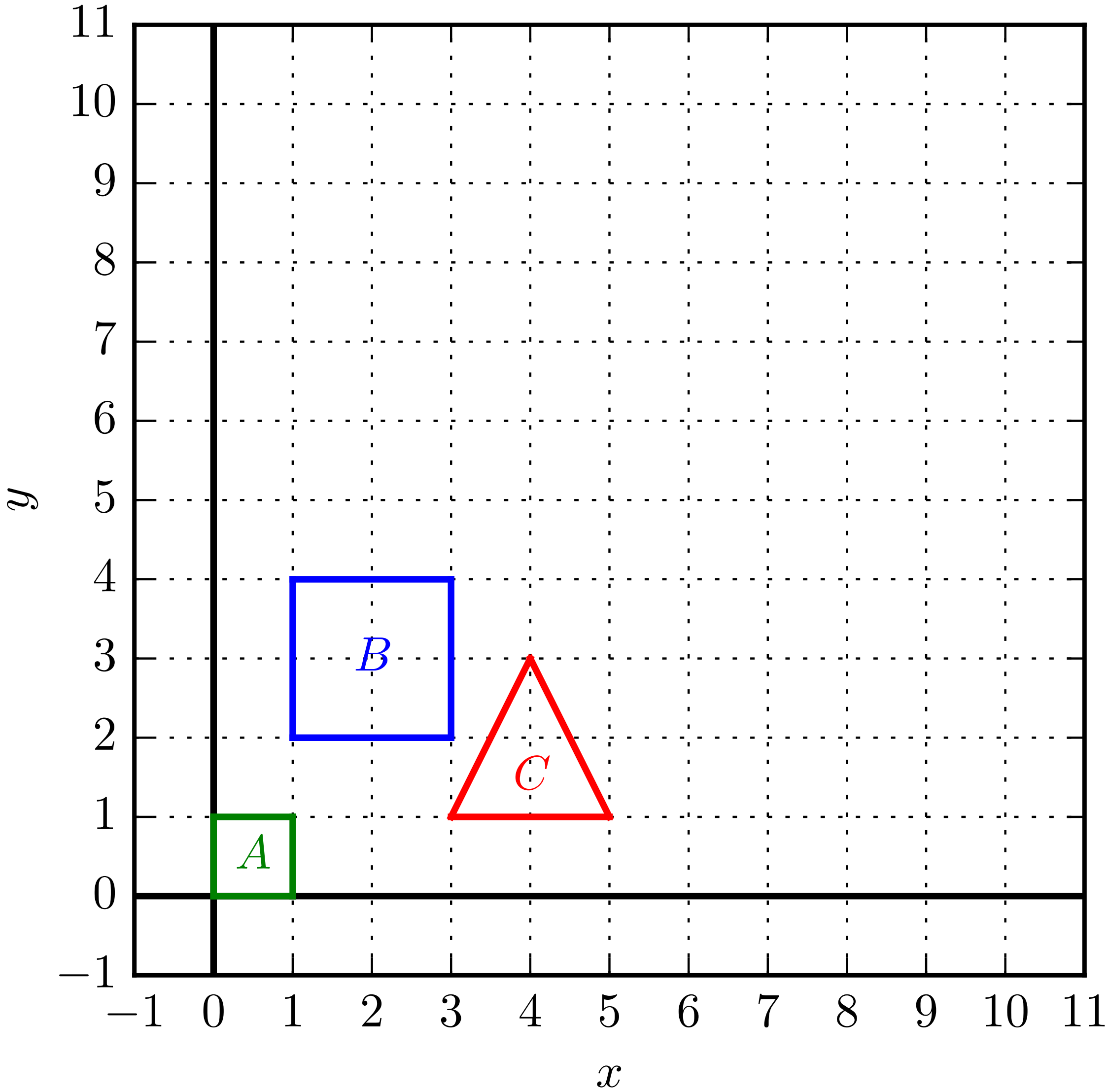
$$M_{(6,1)}S_A =$$

$$M_{(6,1)}S_B =$$

$$M_{(6,1)}S_C =$$

Now consider the matrix $M_{(6,2)} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Now apply the transformation defined by the matrix $M_{(6,2)}$ to the matrices for shapes A, B and C to generate the transformed shapes A'', B'' and C'.

$$M_{(6,2)}S_A =$$



$M_{(6,2)}S_B =$

$M_{(6,2)}S_C =$

Plot, and label, the shapes $A', B', C'A'', B''$ and C'' on the pair of axes on the next page.

Using the answers for the matrices $M_{(6,1)}$ and $M_{(6,2)}$ what geometric transformation does the general matrix,

$$M_6 = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix},$$

represent?

Investigation 7

We wish to investigate the affect of the transformation T_1 which we define by the matrix

$$M_7 = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix},$$

where $k \in \mathbb{R}$.

First consider the matrix $M_{(7,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Now apply the transformation defined by the matrix $M_{(7,1)}$ to the matrices for shapes A, B and C to generate the transformed shapes A', B' and C'.

$$M_{(7,1)}S_A =$$

$$M_{(7,1)}S_B =$$

$$M_{(7,1)}S_C =$$

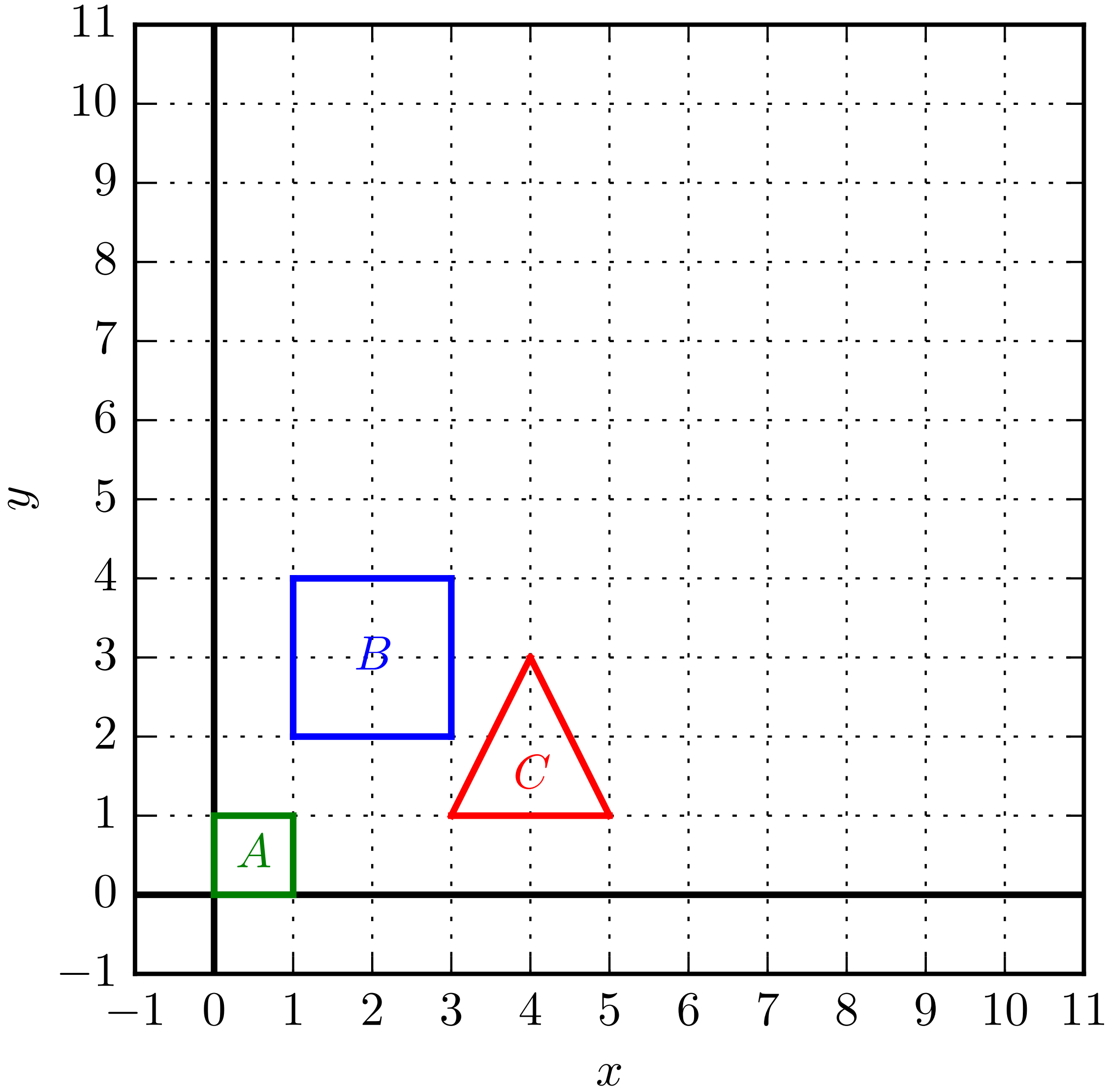
Now consider the matrix $M_{(7,2)} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Now apply the transformation defined by the matrix $M_{(7,2)}$ to the matrices for shapes A, B and C to generate the transformed shapes A'', B'' and C'.

$$M_{(7,2)}S_A =$$

$$M_{(7,2)}S_B =$$

$$M_{(7,2)}S_C =$$

Plot, and label, the shapes A', B', $C'A''$, B'' and C'' on the pair of axes on the next page.



Using the answers for the matrices $M_{(7,1)}$ and $M_{(7,2)}$ what geometric transformation does the general matrix,

$$M_7 = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix},$$

represent?