A - Level Maths Sequences Recap

Arithmetic Sequences	
General form of the sequence	$a,\ a+d,\ a+2d,\ a+3d,$
Term to term rule	add d
nth term rule	$u_n = a+(n-1)d$
Sum of the first n terms	$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+L]$ **Proof:** ~~$S_n = a + \ldots$~~ $S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$ $+\ \ S_n = a+(n-1)d + (a+(n-2)d) + \cdots + (a+2d) + (a+d) + a$ $2S_n = \underbrace{(2a+(n-1)d) + (2a+(n-1)d) + \cdots + (2a+(n-1)d)}_{n}$ So $2S_n = n(2a+(n-1)d)$ $S_n = \frac{n}{2}(2a+(n-1)d)$

Example

Find the sum of the first 10 (n) terms of the arithmetic series with first term 4 (a) and common difference 3 $=d$

$$S_n = \frac{10}{2}[2\times 4 + (9)\times 3]$$

$$= 175$$

Geometric Sequences	
General form of the sequence	$a, ar, ar^2, \ldots$
Term to term rule	$\times$ by r
nth term rule	$u_n = a \times r^{n-1}$
Sum of the first n terms	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$ **Proof:** $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ $rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$ $S_n - rS_n = a - ar^n$ $S_n(1-r) = a(1-r^n)$ $S_n = \frac{a(1-r^n)}{1-r}$
Sum to infinity	$S_\infty = \frac{a}{1-r}, \quad \lvert r \rvert < 1$

Example

Find the sum to infinity of the geometric series with common ratio $\frac{1}{4}$ $= r$ and first term 24. $= a$

$\frac{1}{4} < 1$ then $S_\infty = \frac{a}{1-r} = \frac{24}{1 - 1/4} = \frac{24}{3/4} = 32$

Sequence and Series Properties	
Increasing Sequence	$u_{n+1} > u_n \quad \forall n$ eg: 2, 4, 6, 8, 10 is an increasing sequence
Decreasing Sequence	$u_{n+1} < u_n$ So 30, 25, 20, 15, 10, 5, 0, -5, ... is a decreasing sequence.
Periodic Sequence	$u_n = u_{n+p} \quad \forall n \in \mathbb{N},\ p \in \mathbb{N}$ $u_1\ u_2\ u_3\ u_4\ u_5$ $\underbrace{2\ 2\ 2\ 5}_{4}\ \underbrace{2\ 2\ 2\ 5}_{4}\ 2\ 2\ 2\ 5$ period 4
Sigma Notation	$\sum_{r=1}^{6} (r+1) = 2+3+4+5+6+7$ $= 27$

Example

A sequence is arithmetic with 2nd term 7 and 10th term 31. Find the sum of the first 100 terms.

$$a + d = 7 \quad (1)$$

$$a + 9d = 31 \quad (2)$$

$(2) - (1)$

$$8d = 24$$

$$d = 3 \qquad \Rightarrow a = 4$$

$$S_{100} = \frac{100}{2}\left[2 \times 4 + (99) \times 3\right]$$

$$= 15250$$

Example

A sequence is defined by $u_{n+1} = pu_n + q$ where p and q are constants. The first three terms of the sequence are given by $u_1 = 200$, $u_2 = 100$ and $u_3 = 60$.

a) Find the values of p and q

$u_2 = 100 = p \times u_1 + q$ so $100 = 200p + q$

$\Rightarrow \underbrace{q = 100 - 200p}$

Using $u_3 = pu_2 + q$

$60 = p\,100 + q$

$60 = 100p + 100 - 200p$

$\Rightarrow -40 = -100p$

$p = \frac{4}{10}$

And $q = 100 - 200 \times \frac{4}{10}$

$= 20$

So $u_{n+1} = \frac{4}{10} u_n + 20$

b) Find the value of u_5

$u_1 = 200$

$u_2 = 100$

$u_3 = 60$

$u_4 = \frac{4}{10} \times 60 + 20 = 44$

$u_5 = \frac{4}{10} \times 44 + 20 = 37.6$

c) The limit of u_n as $n \to \infty$ is L. Find L.

In the limit

$L = \frac{4}{10} L + 20$

$\Rightarrow \frac{6L}{10} = 20$ $\Rightarrow L = \frac{100}{3} = 33\frac{1}{3}$