

AQA A-Level Maths 2023 Paper 3

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

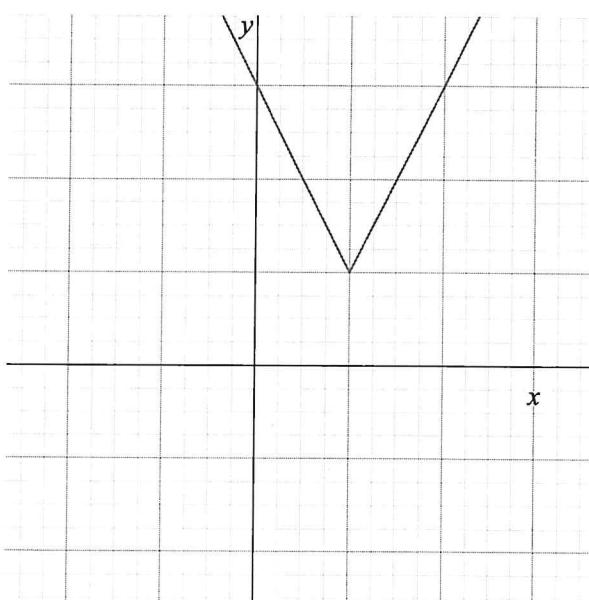
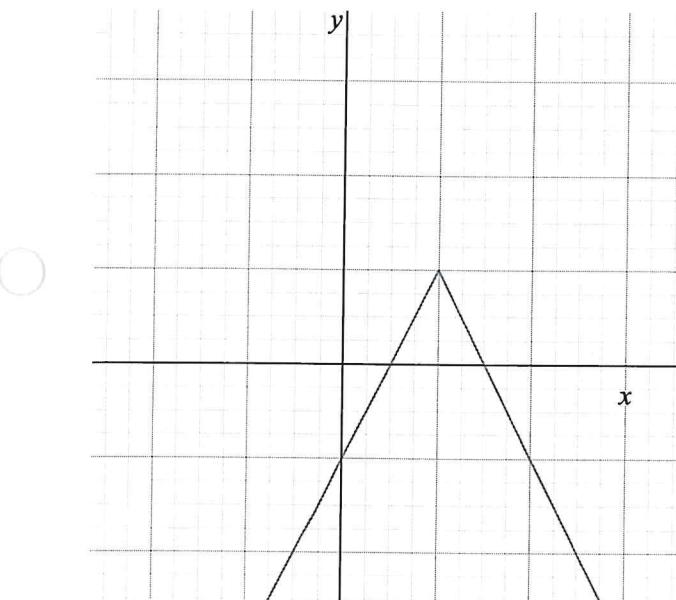
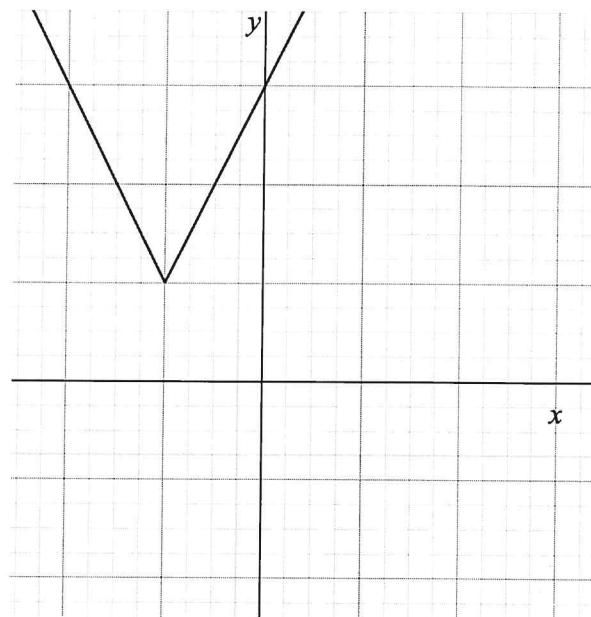
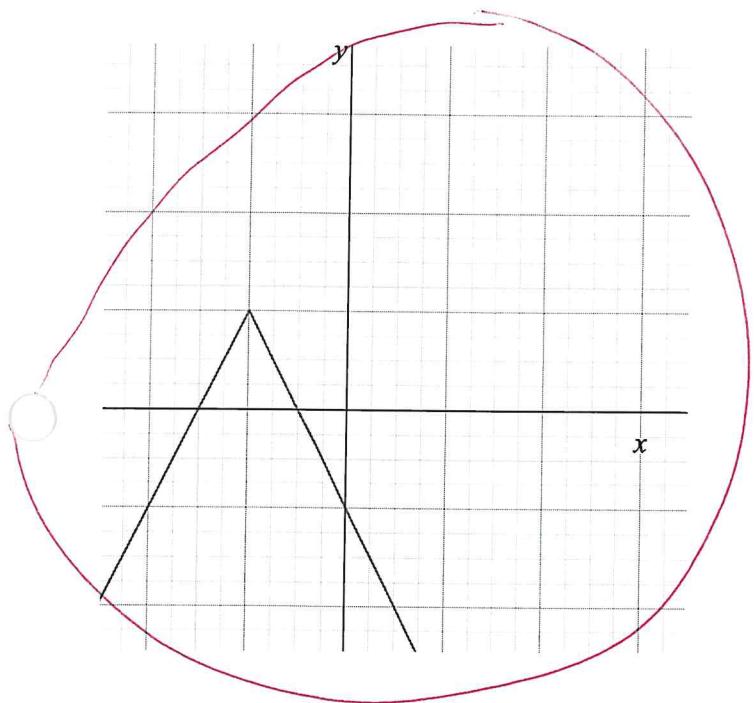
Name: *Solutions*

Total Marks: / 100



SECTION A

- 1 Which of the graphs below is a graph of $y = 2 - |2x + 4|$?



[1 mark]

- 2 How many times do the graphs of $y = \cot(x)$ and $y = \sec(2x)$ intersect in the interval $0 \leq x \leq 2\pi$?

0

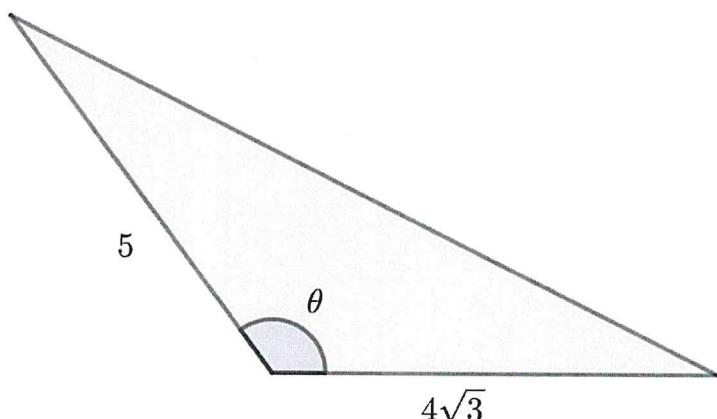
1

2

3

[1 mark]

- 3 The area of the triangle below is 15 square units.



Find the value of θ .

2\pi
3\pi
3\pi
65\pi
6

$$\frac{1}{2} \times 5 \times 4\sqrt{3} \times \sin(\theta) = 15$$

$$\sin(\theta) = \frac{30}{20\sqrt{3}} = \frac{\sqrt{3}}{2}$$

[1 mark]

$$\text{So } \theta = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

- 4 Show that $\sum_{r=1}^5 \ln\left(\frac{r}{r+2}\right) = -\ln(21)$

[3 marks]

$$\begin{aligned}
 \sum_{r=1}^5 \ln\left(\frac{r}{r+2}\right) &= \ln\left(\frac{1}{3}\right) + \ln\left(\frac{2}{4}\right) + \ln\left(\frac{3}{5}\right) + \ln\left(\frac{4}{6}\right) + \ln\left(\frac{5}{7}\right) \\
 &= \ln\left(\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \frac{4}{6} \times \frac{5}{7}\right) \\
 &= \ln\left(\frac{1}{21}\right) \\
 &= \ln(21^{-1}) \\
 &= -\ln(21)
 \end{aligned}$$

- 5 Differentiate, from first principles $f(x) = x^3 + 2x$

[5 marks]

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^3 + 2(x+h) - (x^3 + 2x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 + 2 \\
 &= 3x^2 + 2
 \end{aligned}$$

- 6 The rate of change of the radius of a circle is inversely proportional to the radius cubed.

- a) Show that the rate at which the area of the circle, A , changes

$$\text{satisfies } \frac{dA}{dt} = \frac{2\pi^2}{A}$$

[4 marks]

$$\frac{dr}{dt} = \frac{1}{r^3}$$

$$\text{Dish } \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{r^3}$$

$$\text{Now } A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r$$

$$\Rightarrow \frac{dr}{dA} = \frac{1}{2\pi r}$$

$$\frac{1}{2\pi r} \frac{dA}{dt} = \frac{1}{r^3}$$

$$\Rightarrow \frac{dA}{dt} = \frac{2\pi r}{r^3}$$

$$= \frac{2\pi}{r^2}$$

$$= \frac{2\pi^2}{\pi r^2}$$

$$= \frac{2\pi^2}{A}$$

- b) Explain why $\frac{dA}{dt} > 0$

$$r > 0 \Rightarrow \frac{1}{r^3} > 0$$

[1 mark]

$$\Rightarrow \frac{dr}{dt} > 0$$

$$\Rightarrow \frac{dA}{dt} > 0$$

- 7 Consider the cubic function $p(x) = x^3 + x^2 + kx + 3k$.

- a) Given that $x = 3$ is a root of $p(x)$, find k

$$x = 3 \text{ is a root } \Rightarrow p(3) = 0$$

[2 marks]

$$0 = 27 + 9 + 3k + 3k$$

$$\Rightarrow 6k = -36$$

$$k = -6$$

So

$$p(x) = x^3 + x^2 - 6x - 18$$

- b) Prove that $p(x)$ has only one real root.

[4 marks]

$$p(x) = x^3 + x^2 - 6x - 18$$

$$= (x-3)(x^2 + 4x + 6)$$

Consider the discriminant of $q(x) = x^2 + 4x + 6$

$$4^2 - 4 \times 1 \times 6 = -8$$

Since the discriminant is < 0 there are no real roots of $q(x)$ and so the only real root of $P(x)$ is $x = 3$

- 8 Find the solutions of the equation $\cos(\theta) + 3 \cos\left(\theta + \frac{\pi}{6}\right) = 0$ in the range $0 \leq \theta \leq 2\pi$

[6 marks]

$$\cos(\theta) + 3 \cos\left(\theta + \frac{\pi}{6}\right) = 0$$

$$\cos(\theta) + 3 \left(\cos(\theta) \cos\left(\frac{\pi}{6}\right) - \sin(\theta) \sin\left(\frac{\pi}{6}\right) \right) = 0$$

$$\cos(\theta) + \frac{3\sqrt{3}}{2} \cos(\theta) - \frac{3}{2} \sin(\theta) = 0$$

$$\left(1 + \frac{3\sqrt{3}}{2}\right) \cos(\theta) = \frac{3}{2} \sin(\theta)$$

Hence,

$$\tan(\theta) = \underbrace{\left(1 + \frac{3\sqrt{3}}{2}\right)}_{\frac{1}{2}}$$

$$\Rightarrow \tan(\theta) = \frac{2+3\sqrt{3}}{3}$$

So

$$\theta = \arctan\left(\frac{2+3\sqrt{3}}{3}\right)$$

$$= 1.1758 \quad \text{and} \quad 4.3174$$

9

- a) If $x = 4 \sin(u)$, simplify $\sqrt{16 - x^2}$

[2 marks]

$$\begin{aligned}\sqrt{16 - x^2} &= \sqrt{16 - (4\sin(u))^2} \\ &= \sqrt{16 - 16\sin^2(u)} \\ &= \sqrt{16(1 - \sin^2(u))} \\ &= 4\cos(u)\end{aligned}$$

- b) Hence, or otherwise, find $\int_1^3 \frac{1}{x^2\sqrt{16 - x^2}} dx$.

[8 marks]

Let $u = 4\sin(u)$, then

$$\begin{aligned}\frac{du}{dx} &= 4\cos(u) \quad \Rightarrow dx = 4\cos(u) du \\ &\quad = \sqrt{16 - u^2} du\end{aligned}$$

Hence, consider

$$\begin{aligned}I &= \int \frac{1}{x^2\sqrt{16 - x^2}} dx \\ &= \int \frac{1}{(4\sin(u))^2 \sqrt{16 - 16\sin^2(u)}} \times 4\cos(u) du\end{aligned}$$

$$= \frac{1}{16} \int \csc^2(u) du$$

$$= -\frac{1}{16} \cot(u) + C$$

$$= -\frac{1}{16} \frac{\cos(u)}{\sin(u)}$$

$$\begin{aligned} &= -\frac{1}{16} \frac{\sqrt{16 - x^2}}{x} = -\frac{1}{16} \frac{\frac{1}{4} \sqrt{16 - x^2}}{\frac{1}{4} x} \\ &= -\frac{\sqrt{16 - x^2}}{4x} \end{aligned}$$

Now,

$$\int_1^3 \frac{1}{x^2 \sqrt{16 - x^2}} dx = \frac{1}{16} \left[-\frac{\sqrt{16 - x^2}}{x} \right]_1^3$$

$$= \frac{1}{16} \left[\left(-\frac{\sqrt{15}}{3} \right) - \left(-\frac{\sqrt{15}}{1} \right) \right]$$

$$= \frac{1}{16} \left[\sqrt{15} - \frac{\sqrt{15}}{3} \right]$$

$$= \frac{3\sqrt{15}}{48} - \frac{\sqrt{15}}{48}$$

- 10 Consider the circle $x^2 - 8x + y^2 - 8y + 7 = 0$

a) What is the centre, C , of the circle?

[1 mark]

$$\begin{aligned} x^2 - 8x + y^2 - 8y + 7 &= 0 \\ \Rightarrow (x-4)^2 + (y-4)^2 &= 25 \\ \text{So the centre is } &(4, 4) \end{aligned}$$

b) Find the equation of the tangent to the circle at $P(8, 7)$

[3 marks]

Gradient of $CP = \frac{3}{4} \Rightarrow$ gradient of tangent is $-\frac{4}{3}$

Hence, equation of the tangent is of the form $y = -\frac{4}{3}x + c$, passing through $(8, 7)$.

$$\begin{aligned} 7 &= -\frac{4}{3} \times 8 + c \\ \Rightarrow c &= 7 + \frac{4}{3} \times 8 \end{aligned}$$

$$= \frac{53}{3}$$

Hence, equation of the tangent is

$$y = -\frac{4}{3}x + \frac{53}{3}$$

$$\Rightarrow 3y + 4x = 53$$

- c) Another tangent to the circle at $Q(7,0)$ has equation $3x - 4y = 21$.

This tangent meets the tangent found in (a) at the point R .

Shape T is formed by removing the sector CPQ of the circle from the quadrilateral $CPRQ$.

Find the ratio

$$\text{Perimeter of } T : \text{Area of } T$$

[8 marks]

Solving $3x - 4y = 21$ and
 $3y + 4x = 53$ simultaneously gives the
coordinates of R as $(11, 3)$

$$\text{Gradient } CP = \frac{3}{4}$$

$$\text{Gradient } CQ = -\frac{4}{3}$$

Hence CP and CQ are perpendicular, so
give that CP meets the tangent at 90°
and CQ meets the tangent at 90° , we
must have a quadrilateral with four
 90° angles.

$$\begin{aligned} |PR| &= \sqrt{(11-8)^2 + (3-7)^2} \\ &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

Hence $PR = 5$ also and so we have $CPRQ$ is
a square.

So perimeter of $\triangle PQR$ is 20 and area is 25.

$$\text{Arc length } PQ = S \times \frac{\pi}{2} = \frac{5\pi}{2}$$

$$\begin{aligned}\text{Area of sector } CPQ &= \frac{1}{2} \times S^2 \times \frac{\pi}{2} \\ &= \frac{25\pi}{4}\end{aligned}$$

$$\begin{aligned}\text{Perimeter of } T &= 10 + \frac{5\pi}{2} \\ &= \frac{20 + 5\pi}{2}\end{aligned}$$

$$\begin{aligned}\text{Area of } T &= 2S - \frac{25\pi}{4} \\ &= \frac{2S(4-\pi)}{4}\end{aligned}$$

So the ratios,

$$P:T = \frac{20 + 5\pi}{2} : \frac{2S(4-\pi)}{4}$$

$$2(4+\pi) : S(4-\pi)$$

SECTION B

- 10 The table below shows the probability distribution of a discrete random variable X .

x	1	2	3	4
$P(X = x)$	$2k$	0.4	0.3	$5k$

Find the value of k

$$\frac{3}{70}$$

$$\frac{3}{10}$$

1

$$\frac{3}{20}$$

[1 mark]

$$2k + 0.4 + 0.3 + 5k = 1$$

$$7k = 0.3$$

$$k = \frac{3}{70}$$

$$\mu \quad \sigma^2$$

- 11 Let X be a continuous random variable such that $X \sim N(23, 0.28)$.

Then the standard deviation of X is, correct to 2 decimal places,

0.28

0.53

0.08

6.44

[1 mark]

$$\sigma^2 = 0.28$$

$$\Rightarrow \sigma = \sqrt{0.28}$$

$$= 0.53 \text{ to 2 dp}$$

$$\text{Frequency Density} = \frac{\text{frequency}}{\text{class width}}$$

- 12 A histogram of the following data has been drawn

freq density
→

Range	Frequency	freq Density
$0 \leq x < 20$	32	$\frac{8}{5}$
$20 \leq x < 30$	20	2
$30 \leq x < 35$	x	$\frac{x}{5}$
$35 \leq x < 50$	y	$\frac{y}{15}$

- The height of the first bar, when plotted, is 8 cm and there are 124 data points in total.

- a) The bar representing the range 30 – 35 is 12 cm high. Find x .

$$\text{height} = k \times \text{frequency density}.$$

[2 marks]

So using the equation given

$$8 = k \times \frac{8}{5} \Rightarrow k = 5$$

$$\text{So for } 30-35 \quad 12 = k \times \frac{x}{5}$$

$$12 = 5 \times \frac{x}{5}$$

$$\Rightarrow x = 12$$

- b) Find the frequency density for the range 35 to 50.

[2 marks]

124 data points in total.

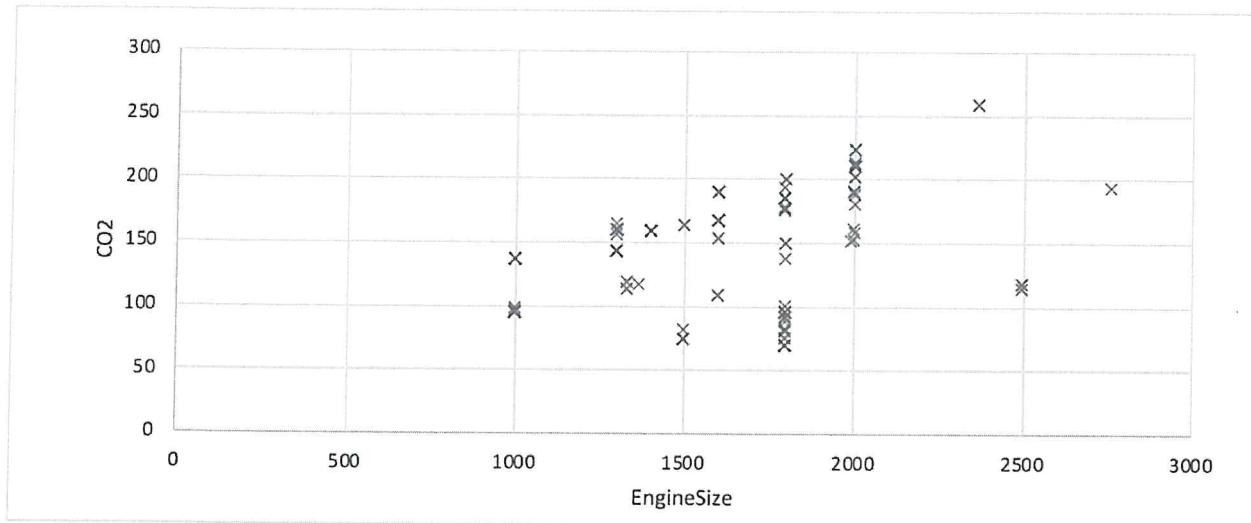
So

$$y = 124 - 32 - 20 - 12 \\ = 60$$

$$\text{So frequency density} = \frac{60}{15} = 4$$

- 13 Miles is using the large data set to investigate whether there is a correlation between engine size and CO_2 emissions.

Looking at a single manufacturer he produces the scatter plot below.



- a) What kind of correlation is shown in the scatter graph above.

[1 marks]

Weak positive

- b) Miles concludes, using data from the large data set, that for all cars in the UK, engine size can be used to predict CO_2 emissions from cars.

Explain why, using your knowledge of the large data set why this may be an unsafe conclusion to make.

[2 marks]

Large data set doesn't contain all cars, and doesn't contain cars from all regions in the UK

- 15 A factory in Germany produces metal ferrules for pencils.

They should have a mean diameter of 7.4 mm.

A sample of 10 ferrules are taken and found to have a sample mean diameter of 6.6 mm.

, 0.05

Is there evidence, at the 5% significance level to suggest that the mean diameter of the ferrules is less than it should be.

You may assume that the population variance is 1 mm².

[5 marks]

Step 1: $H_0: \mu_0 = 7.4$

$H_1: \mu_1 < 7.4$

Assuming that the ferrules are normally distributed,
then

$$\bar{X} \sim N\left(7.4, \frac{1}{10}\right)$$

$$P(\bar{X} \leq 6.6) = 5.7060 \times 10^{-3}$$

$5.7060 \times 10^{-3} < 0.05$, so there is statistically significant evidence to reject H_0 in favour of H_1 and conclude that the ferrules ~~do~~ have a mean diameter less than they should.

- 16 Liala is driving to school in the morning and has to pass through 5 sets of traffic lights on her journey.

Overtime she has estimated that the probability of the lights being against her (i.e. red) is 0.6.

- a) Find the probability of zero sets of the traffic lights being red.

Let X be the number of traffic lights that are red, then, $X \sim B(5, 0.6)$ [1 mark]

$$P(X = 0) = 0.01024$$

- b) Find the probability of there being more than 3 sets of traffic lights on red.

$$\begin{aligned} P(X > 3) &= 1 - P(X \leq 3) && \text{[2 marks]} \\ &= 1 - 0.66304 \\ &= 0.33696 \end{aligned}$$

She estimates that each red light adds 70 seconds to her journey.

- c) Find the expected extra journey time due to waiting at lights.

For a binomial distribution,
expected value is $N \times p$ [2 marks]

So we expect $5 \times 0.6 = 3$ traffic lights to be red.

$$\begin{aligned} \text{So extra time is } 3 \times 70 &= 210 \text{ seconds} \\ &= 3.5 \text{ minutes} \end{aligned}$$

17 A random variable X is normally distributed.

If $P(X \leq 175) = 0.7222$ and $P(X > 190) = 0.1729$, find the mean μ and variance σ^2 , of X

$$\begin{aligned} P(X \leq 175) &= 0.7222 \\ \Rightarrow P\left(Z \leq \frac{175 - \mu}{\sigma}\right) &= 0.7222 \end{aligned}$$

[6 marks]

$$\text{So } \frac{175 - \mu}{\sigma} = 0.5894$$

$$\Rightarrow 175 = \mu + 0.5894\sigma \quad (1)$$

$$P(X > 190) = 0.1729$$

$$\begin{aligned} \Rightarrow P(X < 190) &= 1 - 0.1729 \\ &= 0.8271 \end{aligned}$$

$$\frac{190 - \mu}{\sigma} = 0.9628$$

$$190 = \mu + 0.9628\sigma \quad (2)$$

Solving (1) and (2) simultaneously,

$$\mu = 179.98$$

$$\sigma = 42.44$$

Hence, the variance is 1801.56

- 18 Amir is seeking the views of the Sixth Form on the introduction of a uniform.

There are 260 students in the Sixth Form in total with 140 in Year 12.

He decided to use a random stratified sample to obtain his data.

- a) State one advantage and one disadvantage of stratified sampling.

[2 marks]

Advantage: Produces a representative sample.

Disadvantage: Requiring a list of the entire population

- b) Amir wants a sample of size 65. Determine how many Year 12s and Year 13s Amir should sample.

[3 marks]

$$\text{Yr 12} = \frac{140}{260} \times 65 = 35$$

$$\text{Yr 13} = \frac{120}{260} \times 65 = 30$$

- c) Explain how Amir could collect the random sample of Year 12 students.

[3 marks]

- 1) Give each Year 12 student a random number (integer) from 1 - 160.
- 2) Generate random integers in the range 1 - 160
- 3) Continue until 35 different numbers have been selected and sample these pupils

- d) In practice stratified sampling can be hard to carry out for large populations. What kind of sampling could be used instead?

[1 mark]

Quota sampling

19 A and B are two events such that

$$P(A \cap B) = 0.15$$

$$P(A' \cap B') = 0.4$$

$$P(A) = 2P(B)$$

a) Find $P(A)$

[4 marks]

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\text{and } P(A \cup B) = 1 - P(A' \cap B')$$

$$\approx 0.6$$

$$\begin{aligned} \text{So, } 0.6 &= P(A) + P(B) - P(A \cap B) \\ &= 2P(B) + P(B) - 0.15 \\ \Rightarrow 0.75 &= 3P(B) \end{aligned}$$

$$\begin{aligned} \text{So, } P(B) &\approx 0.25 \\ \text{Hence, } P(A) &= 0.5 \end{aligned}$$

b) Find $P(A | B)$

[2 marks]

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.15}{0.25} \\ &\approx 0.6 \end{aligned}$$

c) Are A and B independent?

[1 mark]

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2}$$

$$\neq 0.15$$

So A and B are not independent.

- 20 a) Where does the maximum of the normal distribution $X \sim N(120, 3^2)$ occur?

[1 mark]

$$\text{When } x = \mu$$

- b) Where do the inflection points of X occur?

$$\text{At } x = \mu \pm \sigma$$

[1 mark]

- b) Let $Y \sim N(\mu, \sigma^2)$. Then Y has probability distribution function

$$f(x) = \frac{1}{\sqrt{2\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

By differentiating find the location of the inflection points of the distribution.

[6 marks]

By the chain rule,

$$f'(x) = -\frac{2(x-\mu)}{2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= -\frac{(x-\mu)}{\sigma^2} f(x)$$

So, using the product rule,

$$u = -\frac{(x-\mu)}{\sigma^2} \quad v = f(x)$$

$$\frac{du}{dx} = -\frac{1}{\sigma^2} \quad \frac{dv}{dx} = -\frac{(x-\mu)}{\sigma^2} f'(x)$$

Hence,

$$f''(x) = -\left(\frac{(x-\mu)}{\sigma^2}\right)' \left(-\frac{(x-\mu)}{\sigma^2} f'(x)\right) - \frac{1}{\sigma^2} f''(x)$$

$$= \frac{f(x)}{\sigma^2} \left(\frac{(x-\mu)^2}{\sigma^2} - 1 \right)$$

at ext point of $f''(x) = 0$

Since $f(x) > 0$, this is zero when

$$(x-\mu)^2 = \sigma^2$$

$$\Rightarrow x = \mu \pm \sigma$$