

AQA A-Level Maths 2023 Paper 1A

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100



1 $\frac{3}{1+\sqrt{2}}$ simplifies to

$3\sqrt{2} - 3$

$3\sqrt{2} + 3$

$\sqrt{2} - 1$

$\sqrt{2} + 1$

$$\frac{3}{1+\sqrt{2}} = \frac{3}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$$

[1 mark]

$$= \frac{3 - 3\sqrt{2}}{1 + \sqrt{2} - \sqrt{2} - 2} = -3 + 3\sqrt{2}$$

- 2 What is the centre of the circle with equation
 $x^2 + 8x + y^2 - 4y - 16 = 0$?

(4,2)

(-4, -2)

(-4,2)

(4, -2)

[1 mark]

$$x^2 + 8x + y^2 - 4y - 16 = 0$$

$$\Rightarrow (x+4)^2 - 16 + (y-2)^2 - 4 = 16 \Rightarrow (x+4)^2 + (y-2)^2 = 36$$

$$\Rightarrow (x+4)^2 + (y-2)^2 = 36$$

- 3 The expansion of $(5 + 4x)^{\frac{1}{2}}$ converges for all x such that

$$|x| \leq \frac{4}{5}$$

$$|x| \leq \frac{5}{4}$$

$$x < \frac{4}{5}$$

$$x < \frac{5}{4}$$

These are corrected

to be <

$$\left| \frac{4+x}{5} \right| \leq 1 \Rightarrow |x| \leq \frac{1}{4}$$

[1 mark]

- 4 Simplify $\frac{x+3}{x^2 - 1} \div \frac{x^2 + 6x + 9}{2x^2 + 5x + 3}$

[3 marks]

$$= \frac{x+3}{(x+1)(x-1)} \times \frac{(x+1)(2x+3)}{(x+3)(x+3)}$$

$$= \frac{2x+3}{(x-1)(x+3)}$$

- 5 Prove that $\log_2 3$ is irrational.

[5 marks]

Suppose for a contradiction that $\log_2(3)$ is rational.

Then $\exists a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ such that

$$\log_2(3) = \frac{a}{b}$$

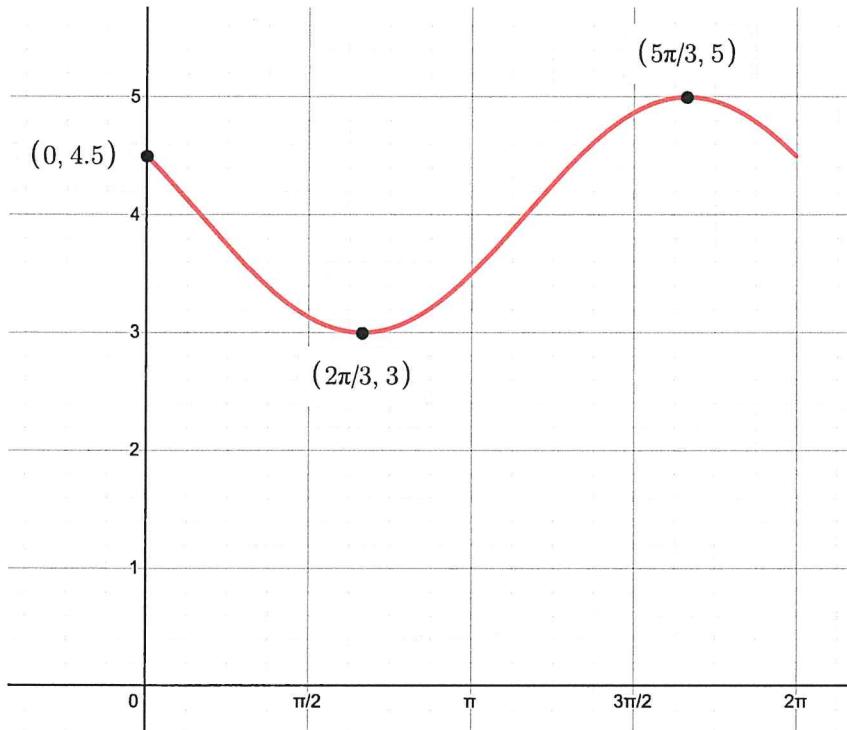
$$\Rightarrow b \log_2(3) = a$$

$$\Rightarrow 2^a = 3^b$$

but 2^a is even & a, 3^b is odd & b. So
 $2^a \neq 3^b$, hence we have a contradiction and
conclude that $\log_2(3)$ is irrational.

- 6 a) Sketch the graph of $y = \cos\left(x + \frac{\pi}{3}\right) + 4$ for $0 \leq x \leq 2\pi$

[3 marks]

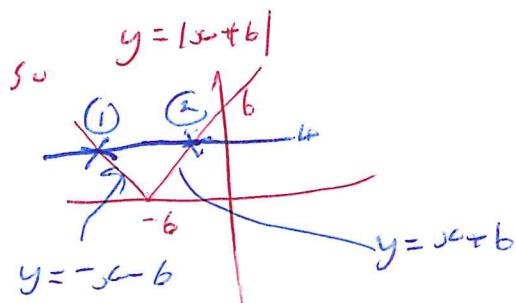
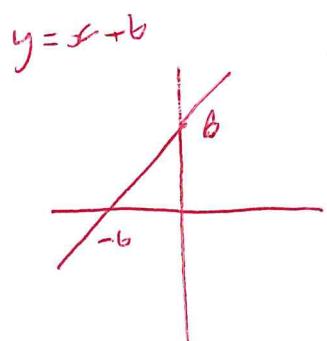


- b) State the coordinates of the minimum point of $y = \cos\left(x + \frac{\pi}{3}\right) + 4$ in the domain $0 \leq x \leq 2\pi$.

[2 marks]

$$\left(\frac{2\pi}{3}, 3\right)$$

- 7 Solve the inequality $|x + 6| < 4$.



[3 marks]

for ①

$$-x - 6 = 4 \Rightarrow -x = 10 \\ x = -10$$

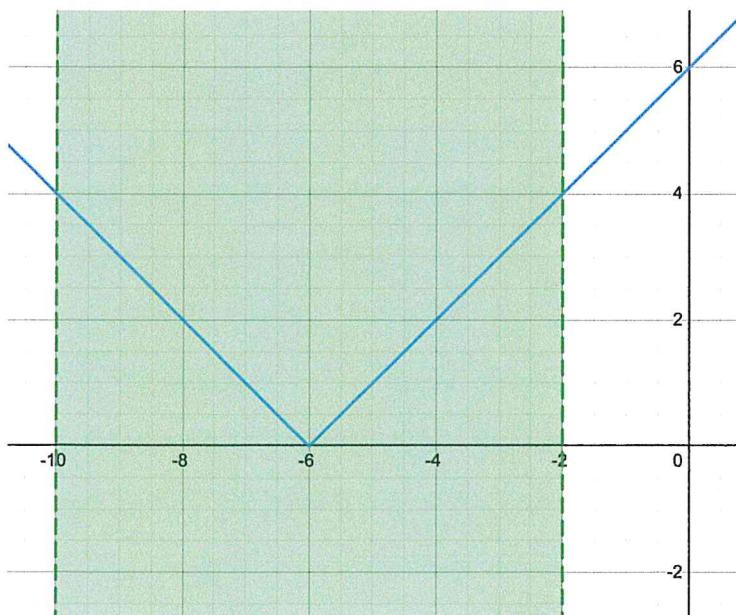
for ②

$$x + 6 = 4 \\ x = -2$$

so solution

$$\{x : -10 < x < -2\}$$

answ $(-10, -2)$



- 8 Find the stationary point of the curve $2x^2y + 6x = 10$.

[6 marks]

$$2x^2y + 6x = 10 \quad (1)$$

Differentiating w.r.t x

$$\begin{aligned} 2x^2 \frac{dy}{dx} + 12xy + 6 &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{6 + 12xy}{2x^2} = -\frac{2 + 4xy}{x^2} \end{aligned}$$

At stationary points $\frac{dy}{dx} = 0$,

$$\Rightarrow -4xy = 6 \quad \Rightarrow y = -\frac{3}{2x}$$

Sub into (1) [must satisfy the original equation too]

$$2x^2 \cdot \frac{-3}{2x} + 6x = 10$$

$$\Rightarrow 3x = 10$$

$$\text{So } x = \frac{10}{3}$$

$$\text{When } x = \frac{10}{3}, y = \frac{-3}{2 \cdot \frac{10}{3}}$$

$$= \frac{-9}{20}$$

$$\text{So the stationary point is } \left(\frac{10}{3}, -\frac{9}{20} \right)$$

- 9 Consider the equation $2x^3 + 3x^2 + 7 = 0$

- a) Show that a solution to the equation above lies in the interval $[-3, -2]$.

[2 marks]

$$f(-3) = -20$$

$$f(-2) = 3$$

Since f is continuous on the interval $[-3, -2]$ and there is a sign change.

- b) Explain why the iterative formula $x_{n+1} = \frac{-7}{2x_n^2 + 3x_n}$, with $x_0 = 1$ fails to find a root of $2x^3 + 3x^2 + 7 = 0$.

[1 mark]

$$\text{Considering } y = \frac{-7}{2x_n^2 + 3x_n}$$

$\rightarrow |(2x_n^2 + 3x_n)| \rightarrow \infty$ as $n \rightarrow \infty$ so it does not converge.

Consider the gradient of y ...

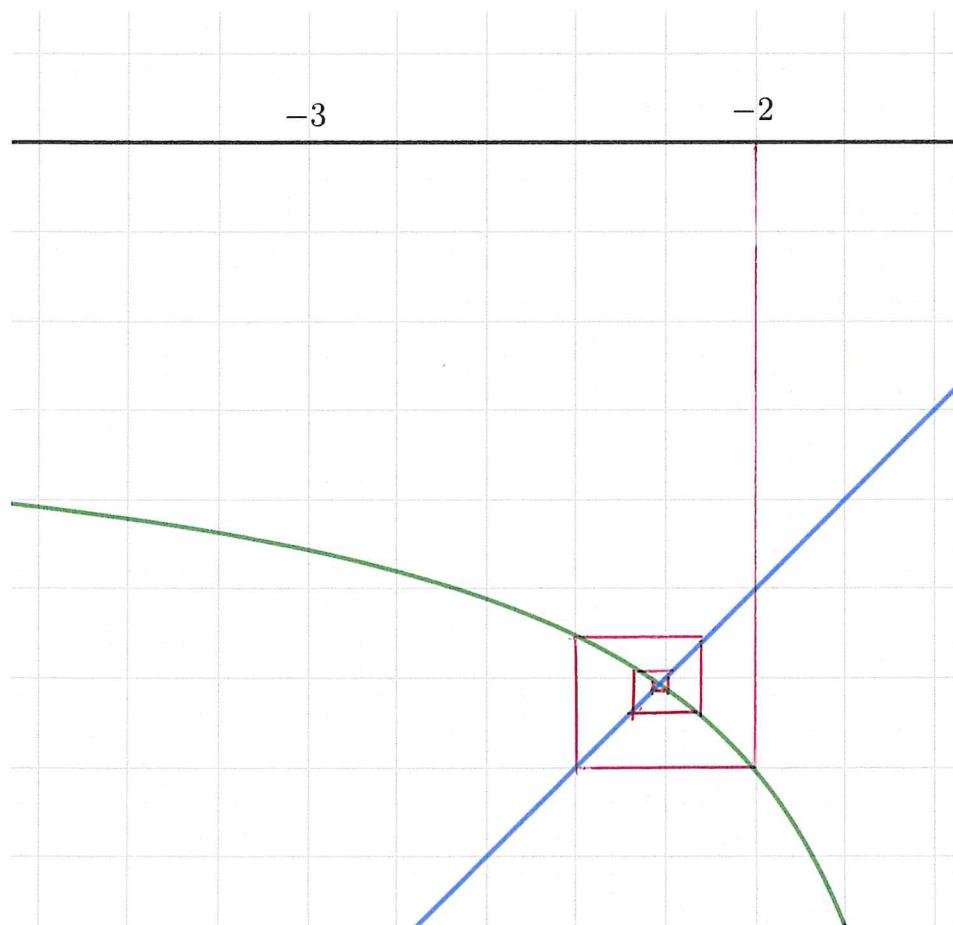
- c) Show that the iterative formula $x_{n+1} = \sqrt[3]{\frac{-7x_n}{2x_n + 3}}$ can be derived from $2x^3 + 3x^2 + 7 = 0$.

[2 marks]

$$\begin{aligned} 2x^3 + 3x^2 &= -7 \\ \Rightarrow x^3 \left(2 + \frac{3}{x}\right) &= -7 \\ x^3 &= \frac{-7}{2 + \frac{3}{x}} \\ \Rightarrow x^3 &= \frac{-7x}{2x + 3} \\ \Rightarrow x &= \sqrt[3]{\frac{-7x}{2x + 3}} \end{aligned}$$

- d) Using the diagram below show that the iterative formula given in (c) with initial value $x_0 = -2$ will converge to a solution of $2x^3 + 3x^2 + 7 = 0$.

[2 marks]



- e) Using the iterative formula given in (c) find the solution to 3 decimal places and show that it is correct to 3dp.

[3 marks]

$$\text{Let } x_0 = -2$$

$$\text{Using } x_{n+1} = \sqrt[3]{\frac{-7x_n}{2x_n + 3}}$$

$$x_1 = -2.410142264$$

$$x_2 = -2.100553821$$

$$x_3 = -2.1304711884$$

$$x_4 = -2.136162177$$

⋮

$$x \approx -2.14015062$$

Calculate $f(-2.135)$ where $f = 2x^3 + 3x^2 + 7$

$$f(-2.135) = 8.296 \times 10^{-3}$$

$$f(-2.145) = -7.83 \times 10^{-3}$$

Hence, as f is cts and there is a sign change the real root of $f(x) = 0$ is -2.14 to 3dp

- 10 The first three terms of a geometric sequence are given by

$$8x + 20$$

$$10x - 8$$

$$2x + 5$$

- a) Show that there are two possible values of x for this to form a geometric sequence.

[4 marks]

$$\frac{10x - 8}{8x + 20} = \frac{2x + 5}{10x - 8}$$

$$\Rightarrow (10x - 8)^2 = (2x + 5)(8x + 20)$$

$$100x^2 - 160x + 64 = 16x^2 + 40x + 160x + 100$$

$$21x^2 - 60x - 9 = 0$$

$$\Rightarrow x = 3, \text{ or } x = -\frac{1}{7}$$

- b) In the case where x is the integer solution found in (a) write down the first term and common ratio of the sequence.

[2 marks]

Let $x = 3$, then the sequence is

44, 22, 11

$$\text{So } a = 44$$

$$r = \frac{1}{2}$$

- c) Find the difference between the sum of the first 5 terms and the sum to infinity of the sequence generated using the values found in (b).

[4 marks]

$$S_5 = \frac{a(1-r^5)}{1-r}$$

$$= \frac{341}{4}$$

$$S_\infty = \frac{a}{1-r}$$

$$= 88$$

$$\text{So the difference} = 88 - \frac{341}{4}$$

$$= \frac{11}{4}$$

- 11 The line $l_1 : y = \frac{3}{4}x + \frac{23}{4}$ is tangent to the circle

$$C_1 : (x - a)^2 + (y - 1)^2 = 25.$$

- a) Given that $a > 0$, find a .

[6 marks]

Consider $(x - a)^2 + (y - 1)^2 = 25 \quad (1)$
 $y = \frac{3}{4}x + \frac{23}{4} \quad (2)$

Sub (2) into (1)

$$(x - a)^2 + \left(\frac{3}{4}x + \frac{23}{4}\right)^2 = 25$$

$$\Rightarrow x^2 - 2ax + a^2 + \frac{9}{16}x^2 + \frac{57}{8}x + \frac{361}{16} = 25$$

$$\Rightarrow \left(1 + \frac{9}{16}\right)x^2 + \left(\frac{57}{8} - 2a\right)x + \left(a^2 - \frac{39}{16}\right) = 0 \quad (3)$$

As a tangent, the discriminant of (3) is required to be zero

Hence

$$\left(\frac{57}{8} - 2a\right)^2 - 4\left(\frac{25}{16}\right)\left(a^2 - \frac{39}{16}\right) = 0$$

$$\Rightarrow -\frac{9}{4}a^2 - \frac{57}{2}a + 66 = 0$$

Hence, $a = 2$ or $a = \frac{-66}{3}$

As $a > 0$, $a = 2$ and the equation of the circle is

$$(x - 2)^2 + (y - 1)^2 = 25$$

- b) Find the equation of the line l_2 which is tangent to the circle C_1 at the point (6,4). **[4 marks]**

Gradient from centre to (6,4) is $\frac{3}{4}$

So gradient of the tangent at (6,4) is $-\frac{4}{3}$

Hence,

$$y = -\frac{4}{3}x + C$$

Passes through (6,4), hence

$$4 = -\frac{4}{3} \times 6 + C \Rightarrow C = 12$$

So equation of tangent is

$$y = -\frac{4}{3}x + 12$$

12 Using the substitution $x = \sec(\theta)$ show that

$$\int \frac{1}{(x^2 - 1)^{\frac{3}{2}}} dx = -\frac{x}{\sqrt{x^2 - 1}} + C$$

[10 marks]

Let $x = \sec(\theta)$

$$\frac{dx}{d\theta} = \sec(\theta) \tan(\theta)$$

and

$$x^2 - 1 = \sec^2(\theta) - 1 = \tan^2(\theta)$$

Hence,

$$I = \int \frac{1}{(\tan^2(\theta))^{\frac{3}{2}}} \sec(\theta) \tan(\theta) d\theta$$

$$= \int \frac{\sec(\theta)}{\tan^2(\theta)} d\theta$$

$$= \int \frac{1}{\cos(\theta)} \frac{\cos^2(\theta)}{\sin^2(\theta)} d\theta$$

$$= \int \frac{\cos(\theta)}{\sin^2(\theta)} d\theta$$

$$= \int \frac{1}{\sin(\theta)} \frac{\cos(\theta)}{\sin(\theta)} d\theta$$

$$= \int \csc(\theta) \cot(\theta) d\theta$$

$$= -\csc(\theta) + C$$

But $\csc(\theta) = \frac{1}{\sin(\theta)}$

$$\text{If } x = \sec(\theta) \Rightarrow \frac{1}{x} = \cos(\theta)$$

$$\text{so } \sin(\theta) = \sqrt{1 - \cos^2(\theta)}$$

$$= \sqrt{1 - \left(\frac{1}{x}\right)^2}$$

$$= \sqrt{\frac{x^2 - 1}{x^2}}$$

hence,

$$I = -\frac{1}{\sqrt{\frac{x^2 - 1}{x^2}}} + C$$

$$= -\frac{x}{\sqrt{x^2 - 1}} + C$$

13 Let $p(x) = 4x^4 + ax^2 + bx - 6$

a) Given that $(x + 2)$ and $(x - 3)$ are factors of $p(x)$ find a and b .

[4 marks]

By the factor theorem,

$$p(-2) = 0$$

$$\Rightarrow 64 + 4a - 2b - 6 = 0$$

$$\Rightarrow -4a + 2b = 58$$

and

$$p(3) = 0, \text{ so}$$

$$81 + 9a + 3b - 6 = 0$$

$$-9a + 3b = 318$$

$$\Rightarrow a = -27 \text{ and } b = -25$$

- b) Hence show that $p(x)$ has a repeated root and find all solutions of the equation $p(x) = 0$.

[4 marks]

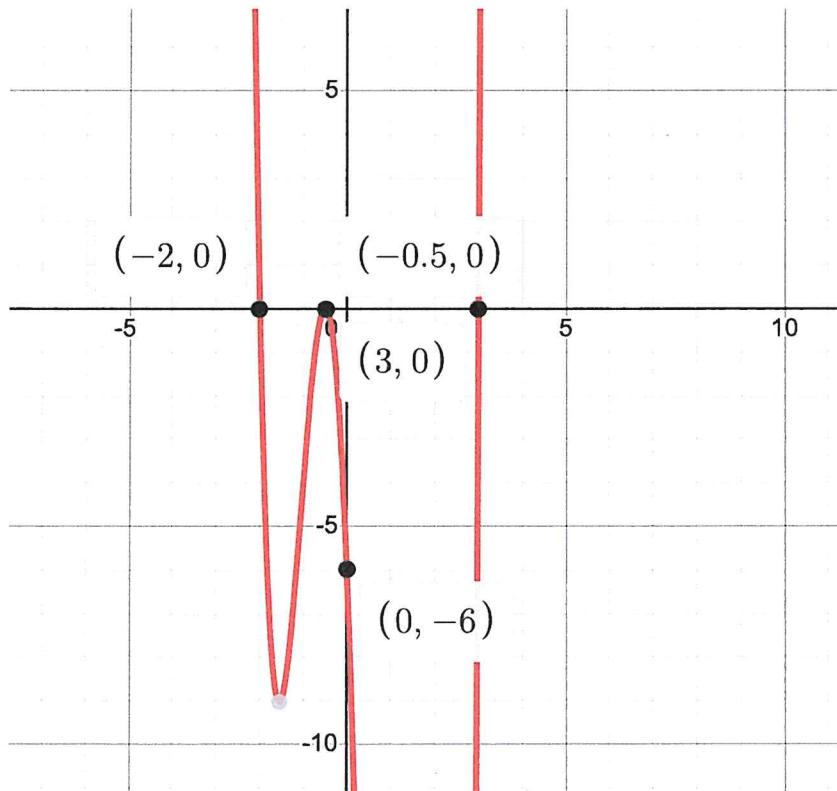
$$\begin{aligned}
 4x^4 + ax^2 + bx - 6 &= (x+2)(x-3)(ax^2 + bx + c) \\
 &\quad \uparrow \qquad \uparrow \\
 &= (-2) \qquad (-2) \\
 &= (x^2 - x - 6)(4x^2 + 4x + 1) \\
 &= (x^2 - x - 6)(2x + 1)^2 \\
 &= (x - 3)(x + 2)(2x + 1)^2
 \end{aligned}$$

So there is a repeated root $x = -\frac{1}{2}$.

Other roots are $x = 3$ and $x = -2$

- c) Sketch $p(x)$ labelling all intersections of the graph with the axes.

[2 marks]



d) Using (c), or otherwise, show that the equation

$$4 \sin^4(\theta) + 19 \sin^2(\theta) - 25 \cos(\theta) - 29 = 0$$

has only two solutions in the range $0 \leq \theta \leq 2\pi$.

[5 marks]

$$4 \sin^4(\theta) + 19 \sin^2(\theta) - 25 \cos(\theta) - 29 = 0$$

$$\Rightarrow 4(1 - \cos^2 \theta)^2 + 19(1 - \cos^2 \theta) - 25 \cos(\theta) - 29 = 0$$

$$\Rightarrow 4(1 - 2\cos^2 \theta + \cos^4 \theta) + 19 - 19\cos^2 \theta - 25 \cos(\theta) - 29 = 0$$

$$\Rightarrow 4 - 8\cos^2 \theta + 4\cos^4 \theta + 19 - 19\cos^2 \theta - 25 \cos(\theta) - 29 = 0$$

$$\Rightarrow 4\cos^4 \theta - 27\cos^2 \theta - 25 \cos(\theta) - 6 = 0 \quad \textcircled{1}$$

Let $x = \cos \theta$, then (1) becomes

$$4x^4 - 27x^2 - 25x - 6 = 0$$

$$\Rightarrow (x-3)(x+2)(2x+1) = 0 \quad \text{using (4)}$$

$\therefore \cos(\theta) = 3$ no solution

$\cos(\theta) = -2$ no solution

or

$$\cos(\theta) = -\frac{1}{2}$$

$$\Rightarrow \theta = \arccos\left(-\frac{1}{2}\right) \quad \text{[principal root]}$$

So

$$\theta = \frac{2\pi}{3} \text{ and } \frac{4\pi}{3}$$

- 14 a) Express in partial fractions $\frac{7x+9}{2x^2+5x+3}$

[3 marks]

Consider

$$\frac{7x+9}{2x^2+5x+3} = \frac{A}{x+1} + \frac{B}{2x+3}$$

So,

$$7x+9 = A(2x+3) + B(x+1)$$

When $x = -1$

$$2 = A$$

When $x = -\frac{3}{2}$

$$-\frac{3}{2} = -\frac{1}{2}B \Rightarrow B = 3$$

Hence,

$$\frac{7x+9}{2x^2+5x+3} = \frac{2}{x+1} + \frac{3}{2x+3}$$

- b) Hence, find a series expansion for $y = \frac{7x+9}{2x^2+5x+3}$ in increasing powers of x , up to the term involving x^3 .

$$\frac{1}{x+1} = (x+1)^{-1}$$

$$= 1 - x + x^2 - x^3$$

$$\frac{1}{2x+3} = (2x+3)^{-1}$$

$$= 3^{-1} \left(1 + \frac{2x}{3}\right)^{-1}$$

$$= \frac{1}{3} \left[1 - \frac{2x}{3} + \frac{4x^2}{9} - \frac{8x^3}{27} \right]$$

$$= \frac{1}{3} - \frac{2x}{9} + \frac{4x^2}{27} - \frac{8x^3}{81}$$

 \therefore

$$\frac{2}{x+1} + \frac{3}{2x+3} = 2\left(1 - x + x^2 - x^3\right) + 3\left(\frac{1}{3} - \frac{2x}{9} + \frac{4x^2}{27} - \frac{8x^3}{81}\right)$$

$$= 3 - \frac{8x}{3} + \frac{22x^2}{9} - \frac{62x^3}{27}$$

- c) John uses this series expansion to find an approximate value,
 I_{approx} of

$$I = \int_0^1 \frac{7x + 9}{2x^2 + 5x + 3} dx$$

Find this approximate value to 4 decimal places.

[2 marks]

$$\begin{aligned} \int_0^1 \frac{7x+9}{2x^2+5x+3} dx &\approx \int_0^1 \left[3 - \frac{8}{3}x + \frac{22}{9}x^2 - \frac{62}{27}x^3 \right] dx \\ &= \left[3x - \frac{4}{3}x^2 + \frac{22}{27}x^3 - \frac{31}{54}x^4 \right]_0^1 \\ &= \frac{103}{54} \end{aligned}$$

d) By first showing that $\int_0^1 \frac{7x+9}{2x^2+5x+3} dx = \frac{1}{2} \ln\left(\frac{a}{27}\right)$

where a is an integer to be found, find the percentage error made by John in his approximation.

[7 marks]

$$\begin{aligned}
 \int_0^1 \frac{7x+9}{2x^2+5x+3} dx &= \int_0^1 \frac{2}{x+1} + \frac{3}{2x+3} dx \\
 &= \left[2 \ln|x+1| + \frac{3}{2} \ln|2x+3| \right]_0^1 \\
 &= 2 \ln(2) + \frac{3}{2} \ln(3) - \left(2 \ln(1) + \frac{3}{2} \ln(3) \right) \\
 &= \ln(4) + \frac{1}{2} \ln(125) - \frac{1}{2} \ln(27) \\
 &= \frac{1}{2} \ln(16) + \frac{1}{2} \ln(125) - \frac{1}{2} \ln(27) \\
 &= \frac{1}{2} \ln\left(\frac{2000}{27}\right)
 \end{aligned}$$

∴

$$\begin{aligned}
 \% \text{ error} &= \left(\frac{\frac{1}{2} \ln\left(\frac{2000}{27}\right) - \frac{103}{54}}{\frac{1}{2} \ln\left(\frac{2000}{27}\right)} \right) \times 100
 \end{aligned}$$

$\approx 17\%$