

AQA A-Level Further Maths 2023 Paper

2B

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

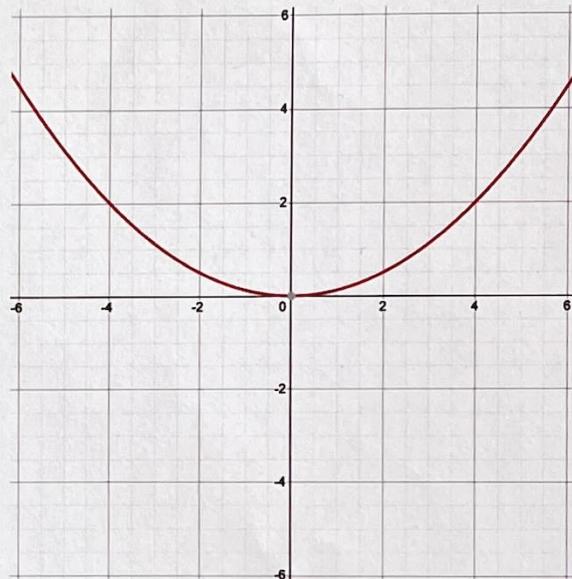
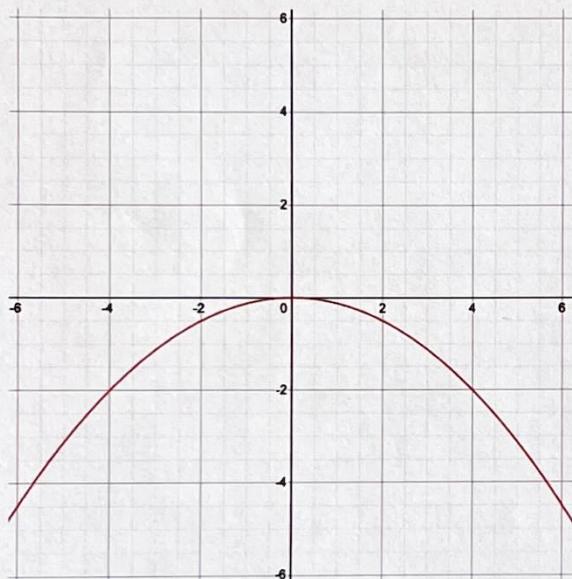
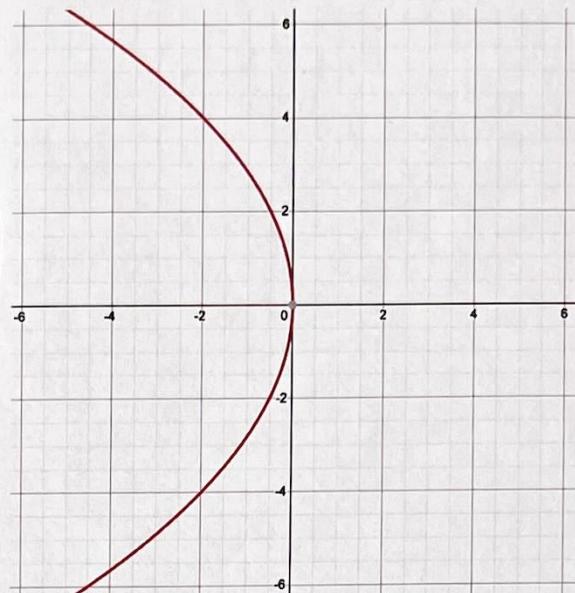
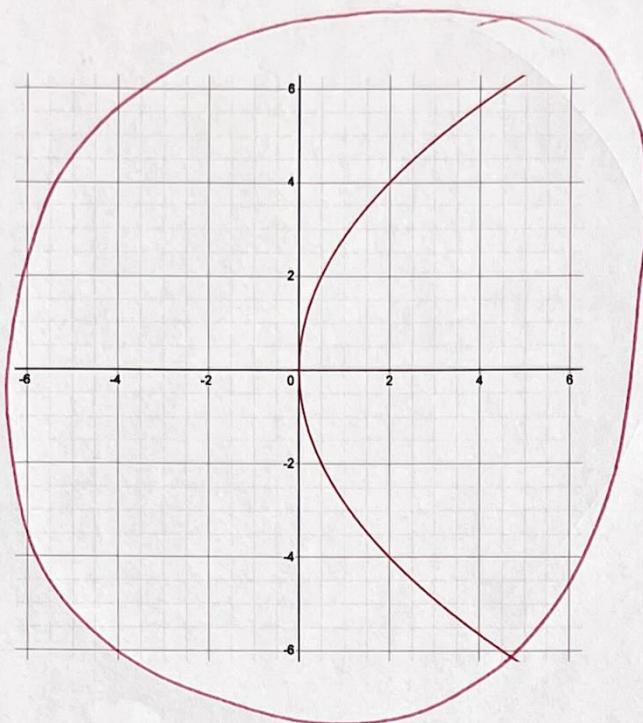
Name:

Total Marks: / 100

(1 mark)



- 1 Which of the following is a plot of $y^2 = 8x$



[1 mark]

- 2 The volume of revolution generated when the region between the curve $y = \ln(x)$, the y -axis and the lines $y = 1$ and $y = 2$ can be computed by evaluating:

$$\pi \int_1^3 e^{2y} dy$$

$$\pi \int_1^3 e^y dy$$

$$\pi \int_1^3 \ln(y) dy$$

$$\pi \int_1^3 (\ln(y))^2 dy$$

[1 mark]

- 3 In the form $a + bi$, $\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right)^4$ is

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$-\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

[1 mark]

- 4 Find the length of the arc of the curve with equation $y = \cosh(x)$ from the point A where $x = 0$ to the point B where $x = \ln(2)$.

[5 marks]

$$y = \cosh(x)$$

$$\frac{dy}{dx} = \sinh(x)$$

$$\text{Hence, } 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2(x) = \cosh^2(x)$$

$$\begin{aligned} \text{Arc length} &= \int_0^{\ln(2)} \sqrt{\cosh^2(x)} \, dx \\ &= \int_0^{\ln(2)} \cosh(x) \, dx \\ &= [\sinh(x)]_0^{\ln(2)} \\ &= \frac{e^{\ln(2)} - e^{-\ln(2)}}{2} \\ &= \frac{2 - \frac{1}{2}}{2} \\ &= \frac{3}{4} \end{aligned}$$

- 5 a) Starting from the exponential definitions of $\sinh(x)$ and $\cosh(x)$
show that $\sinh(2x) = 2 \sinh(x) \cosh(x)$

[3 marks]

Consider the RHS

$$\begin{aligned}
 2 \sinh(x) \cosh(x) &= 2 \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) \\
 &= \frac{1}{2} \left[\frac{e^{2x} - 1 + 1 - e^{-2x}}{4} \right] \\
 &= \frac{1}{2} (e^{2x} - e^{-2x}) \\
 &= \sinh(2x)
 \end{aligned}$$

as required

- b) Hence, or otherwise, obtain an identity for $\tanh(2x)$ in terms of $\tanh(x)$. You may use without proof that

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

[4 marks]

$$\begin{aligned}
 \tanh(2x) &= \frac{\sinh(2x)}{\cosh(2x)} \\
 &= \frac{2 \sinh(x) \cosh(x)}{\cosh^2(x) + \sinh^2(x)} \\
 &= \frac{2 \sinh(x) + \cosh(x)}{\cosh^2(x)} \\
 &\quad \frac{\cosh^2(x)}{\cosh^2(x)} + \frac{\sinh^2(x)}{\cosh^2(x)} \\
 &= \frac{2 \tanh(x)}{1 + \tanh^2(x)}
 \end{aligned}$$

- 6 Prove by induction that the sequence defined by the recurrence formula, with $n \in \mathbb{Z}^+$,

$$f(n+1) = 2f(n) + 3, \quad n \geq 1, \quad f(1) = 1$$

has the closed form solution $f(n) = 2^{n+1} - 3$.

Let $P(n)$ be the statement $\Rightarrow f(n) = 2^{n+1} - 3$

Step 1: From $f(n) = 2^{n+1} - 3$ we have $f(1) = 2^2 - 3 = 1$ which is our given first term

Step 2: We assume $f(k) = 2^{k+1} - 3$ holds

[7 marks]

Step 3: $f(k+1) = 2f(k) + 3$ by recurrence formula

$$= 2(2^{k+1} - 3) + 3 \text{ by inductive hypothesis}$$

$$= 2^{k+2} - 6 + 3$$

$$= 2^{k+2} - 3$$

Hence if $P(k)$ is true then so is $P(k+1)$

Step 4: Since the result is true for $n=1$, and if true for $n=k$ it is also true for $n=k+1$, it is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

- 7 a) For the equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$, where t is time, x is the displacement from the initial position at time t , v is the velocity at time t and a is the amplitude of the resulting simple harmonic motion, show that

$$v^2 = \omega^2(a^2 - x^2)$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

[4 marks]

$$\Rightarrow v \frac{dv}{dx} = -\omega^2 x$$

$$\text{so. } \int v dv = - \int \omega^2 x dx$$

$$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + C$$

When $x=a$, $v=0$, so

$$\frac{0^2}{2} = -\frac{\omega^2 a^2}{2} + C \Rightarrow C = \frac{\omega^2 a^2}{2}$$

$$\text{so. } \frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 a^2}{2}$$

$$\Rightarrow v^2 = \omega^2(a^2 - x^2)$$

- b) Using (a) determine the maximum speed of a particle undergoing simple harmonic motion defined by the equation $\frac{d^2x}{dt^2} = -25x$ where the amplitude of the motion is 4m.

[2 marks]

$$\begin{aligned}v^2 &= 25(4^2 - 0) \\&= 400 \\ \Rightarrow v &= 20 \text{ m s}^{-1}\end{aligned}$$

8 Consider $\begin{vmatrix} x & 1 & 2 \\ x^2 & 1 & 4 \\ x^3 & 1 & 8 \end{vmatrix}$

a) Show that x is a factor of the determinant given above.

[2 marks]

$$\begin{vmatrix} x & 1 & 2 \\ x^2 & 1 & 4 \\ x^3 & 1 & 8 \end{vmatrix} = x \begin{vmatrix} 1 & 1 & 2 \\ x & 1 & 4 \\ x^2 & 1 & 8 \end{vmatrix}$$

b) Factorise $\begin{vmatrix} x & 1 & 2 \\ x^2 & 1 & 4 \\ x^3 & 1 & 8 \end{vmatrix}$ completely.

[5 marks]

$$x \begin{vmatrix} 1 & 1 & 2 \\ x & 1 & 4 \\ x^2 & 1 & 8 \end{vmatrix} = 2x \begin{vmatrix} 1 & 1 & 1 \\ x & 1 & 2 \\ x^2 & 1 & 4 \end{vmatrix}$$

$$= 2x \begin{vmatrix} 0 & 1 & 1 \\ x-1 & 1 & 2 \\ x^2-1 & 1 & 4 \end{vmatrix}$$

$$= 2x \begin{vmatrix} 0 & 1 & 1 \\ x-1 & 1 & 2 \\ (x-1)(x+1) & 1 & 4 \end{vmatrix}$$

$$= 2x(x-1) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ x+1 & 1 & 4 \end{vmatrix}$$

$$= 2x(x-1) \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ x+1 & 1 & 3 \end{vmatrix}$$

$$= 2x(x-1) \left(-1 \begin{vmatrix} 1 & 1 \\ x+1 & 3 \end{vmatrix} \right)$$

$$= 2x(x-1)(x-2)$$

- 9 a) Find the oblique asymptote for the curve with equation

$$y = \frac{(x+3)(x+2)}{x+1}$$

$$\begin{array}{r} xc + 4 \\ x+1) \overline{xc^2 + 5x + 6} \\ xc^2 + x \\ \hline 4x + 6 \\ 4x + 4 \\ \hline 2 \end{array}$$

[3 marks]

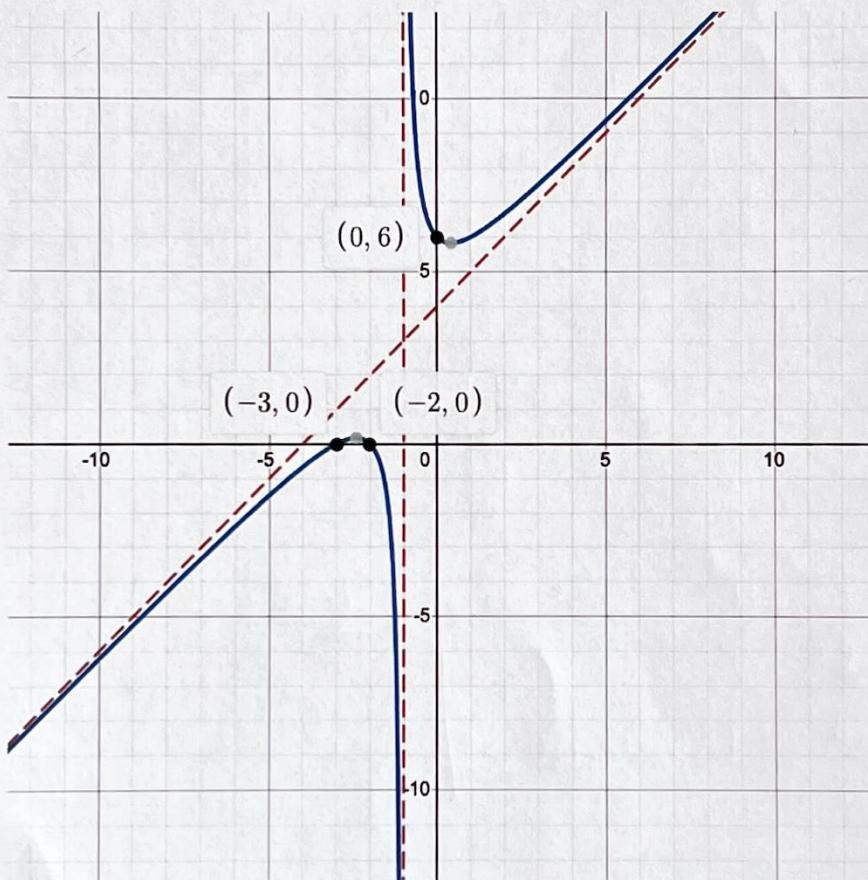
So

$$y = xc + 4 + \frac{2}{x+1}$$

Hence, oblique asymptote $y = xc + 4$

- b) Sketch $y = \frac{(x+3)(x+2)}{x+1}$

[3 marks]



10 Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{2(x+2)}{x(x-1)(x^2+5)}$$

$$I.F. = e^{\int \frac{2}{x} dx} = x^2 \quad [10 \text{ marks}]$$

Hence,

$$x^2 \frac{dy}{dx} + 2x^2 y = \frac{2x(x+2)}{(x-1)(x^2+5)}$$

$$\Rightarrow \frac{d}{dx} \left[x^2 y \right] = \frac{2x^2 + 4x}{(x-1)(x^2+5)}$$

Consider

$$\frac{2x^2 + 4x}{(x-1)(x^2+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+5}$$

$$\begin{aligned} \Rightarrow 2x^2 + 4x &= Ax^2 + SA + (Bx+C)(x-1) \\ &= Ax^2 + SA + Bx^2 - Bx + Cx - C \end{aligned}$$

$$\boxed{x^2} : 2 = A + B$$

$$\boxed{x} : 4 = -B + C$$

$$\boxed{c} : 0 = SA - C$$

So

$$A = 1$$

$$B = 1$$

$$C = S$$

Hence,

$$\frac{d}{dx} \left[x^2 y \right] = \frac{1}{x-1} + \frac{2x+S}{x^2+5}$$

so,

$$\begin{aligned}x^2y &= \int \frac{1}{x-1} + \frac{sx+s}{x^2+s} dx \\&= \ln|x-1| + \int \frac{x}{x^2+s} + \frac{s}{x^2+s} dx + C \\&= \ln|x-1| + \frac{1}{2} \ln|x^2+s| + \sqrt{s} \arctan\left(\frac{x}{\sqrt{s}}\right) + C\end{aligned}$$

So, the general solution is

$$y = \frac{1}{x^2} \left[\ln|x-1| + \frac{1}{2} \ln|x^2+s| + \sqrt{s} \arctan\left(\frac{x}{\sqrt{s}}\right) + C \right]$$

11 By considering the 5th roots of -1 , prove that

$$\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) = \frac{1}{2}$$

[10 marks]

$$-1 = \cos(\pi) + i\sin(\pi)$$

Hence, considering

$$(\cos(\theta) + i\sin(\theta))^5 = \cos(\pi) + i\sin(\pi)$$

$$\Rightarrow \cos(5\theta) + i\sin(5\theta) = \cos(\pi) + i\sin(\pi)$$

So,

$$5\theta = \pi + 2n\pi \Rightarrow \theta = \frac{(2n+1)\pi}{5}$$

Values of $\frac{\pi}{5} \leq \theta \leq \frac{11\pi}{5}$ give the distinct values, so

$$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5} \text{ and } \frac{9\pi}{5}$$

Hence, the 5 roots of -1 are

$$z_1 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$$

$$z_2 = \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)$$

$$z_3 = \cos(\pi) + i\sin(\pi) = -1$$

$$z_4 = \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right) = \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)$$

$$z_5 = \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right) = \cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)$$

Now, sum of complex roots of any number is zero, i.e.

$$Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 0$$

$$\Rightarrow \operatorname{Re}(Z_1 + Z_2 + Z_3 + Z_4 + Z_5) = 0$$

Hence,

$$\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) + (-1) + \underbrace{\cos\left(-\frac{\pi}{5}\right)}_{=\cos\left(\frac{\pi}{5}\right)} + \underbrace{\cos\left(-\frac{3\pi}{5}\right)}_{=\cos\left(\frac{3\pi}{5}\right)} = 0$$

$$\Rightarrow 2\cos\left(\frac{\pi}{5}\right) + 2\cos\left(\frac{3\pi}{5}\right) - 1 = 0$$

i.e.

$$\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = \frac{1}{2}$$

- 12 Three points in three dimensional space have coordinates $A(1,1,1)$, $B(2,2,2)$ and $C(1,0,3)$.

a) Find the equation of the plane, Π_1 , containing A , B and C .

[4 marks]

$$\vec{AB} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{OC} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 0$$

So equation of the plane is

$$5 \cdot \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 0$$

or

$$3x - 2y - z = 0$$

- b) What does this mean about the plane and the origin?

[1 mark]

The plane goes through the origin.

- c) Find the distance from the point $D(4,2,1)$ to the plane Π_1 .

Let L_1 be the line perpendicular to Π_1 , passing through D , then

$$L_1: \vec{r} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

L_1 intersects Π_1 , when

$$\begin{pmatrix} 4+3\lambda \\ 2-2\lambda \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 0$$

$$\Rightarrow 12 + 9\lambda - 4 + 4\lambda - 1 + \lambda = 0$$

$$14\lambda = -7$$

$$\lambda = -\frac{1}{2}$$

Hence the intersection point is

$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 3 \\ \frac{3}{2} \end{pmatrix}$$

So the distance from D to the plane is

$$\sqrt{(4 - \frac{5}{2})^2 + (2 - 3)^2 + (1 - \frac{3}{2})^2} = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$$

- d) Find the coordinates of the reflection of D in Π_1 .

[4 marks]

From (c) the intersection point is $\begin{pmatrix} \frac{5}{2} \\ 3 \\ \frac{3}{2} \end{pmatrix}$

Then

$$\vec{EP'} = \vec{DE}$$

$$\underline{d}' - \underline{e} = \underline{e} - \underline{d}$$

$$\Rightarrow \underline{d}' = 2\underline{e} - \underline{d}$$

$$= 2 \begin{pmatrix} \frac{5}{2} \\ 3 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

So the coordinates of the ~~point of intersection~~ reflection of D in Π_1 are $(1, 4, 2)$

- 13 By considering $\sum_{r=1}^n (\cos(r\theta) + i \sin(r\theta))$ show that

$$\sum_{r=1}^n \sin(r\theta) = \frac{\sin\left(\frac{(n+1)\theta}{2}\right) \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

[14 marks]

$$\begin{aligned} \text{Let } S &= \sum_{r=1}^n \cos(r\theta) + i \sin(r\theta) \\ &= \sum_{r=1}^n (\cos(\theta) + i \sin(\theta))^r \quad \text{by De Moivre} \\ &= (\cos(\theta) + i \sin(\theta)) + (\cos(\theta) + i \sin(\theta))^2 + \cdots + (\cos(\theta) + i \sin(\theta))^n \end{aligned}$$

This is a geometric series with $a = \cos(\theta) + i \sin(\theta)$
and $r = \cos(\theta)$

Consider

$$\begin{aligned} 1 - r^n &= 1 - (\cos(\theta) + i \sin(\theta))^n \\ &= 1 - (\cos(n\theta) + i \sin(n\theta)) \\ &= 2 \sin^2\left(\frac{n\theta}{2}\right) - 2i \sin\left(\frac{n\theta}{2}\right) \cos\left(\frac{n\theta}{2}\right) \end{aligned}$$

$$\text{Similarly, } 1 - r = 2 \sin\left(\frac{\theta}{2}\right) \left[\sin\left(\frac{\theta}{2}\right) - i \cos\left(\frac{\theta}{2}\right) \right]$$

$$1 - r = 2 \sin\left(\frac{\theta}{2}\right) \left[\sin\left(\frac{\theta}{2}\right) - i \cos\left(\frac{\theta}{2}\right) \right]$$

Hence,

$$\frac{1 - r^n}{1 - r} = \frac{2 \sin\left(\frac{n\theta}{2}\right) \left[\sin\left(\frac{n\theta}{2}\right) - i \cos\left(\frac{n\theta}{2}\right) \right]}{2 \sin\left(\frac{\theta}{2}\right) \left[\sin\left(\frac{\theta}{2}\right) - i \cos\left(\frac{\theta}{2}\right) \right]}$$

$$\begin{aligned}
 &= \frac{\sin\left(\frac{n\alpha}{2}\right) \left[\sin\left(\frac{n\alpha}{2}\right) - i\cos\left(\frac{n\alpha}{2}\right) \right]}{\sin\left(\frac{\alpha}{2}\right) \left[\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right) \right]} \times \frac{\sin\left(\frac{\alpha}{2}\right) + i\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right) + i\cos\left(\frac{\alpha}{2}\right)} \\
 &= \frac{\sin\left(\frac{n\alpha}{2}\right) \left[\sin\left(\frac{n\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{n\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{n\alpha}{2}\right) + \cos\left(\frac{n\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) \right]}{\sin\left(\frac{\alpha}{2}\right) \left[\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right) \right]} \\
 &= \frac{\sin\left(\frac{n\alpha}{2}\right) \left[\left(\sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{n\alpha}{2}\right) + \cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{n\alpha}{2}\right) \right) + i \left(\sin\left(\frac{n\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{n\alpha}{2}\right) \right) \right]}{\sin\left(\frac{\alpha}{2}\right)} \\
 &= \frac{\sin\left(\frac{n\alpha}{2}\right) \left[\cos\left(\left(\frac{n-1}{2}\right)\alpha\right) + i\sin\left(\left(\frac{n-1}{2}\right)\alpha\right) \right]}{\sin\left(\frac{\alpha}{2}\right)}
 \end{aligned}$$

 \therefore

$$\sum_{r=1}^n \cos(r\alpha) + i\sin(r\alpha) = \frac{(\cos\alpha + i\sin\alpha) \sin\left(\frac{n\alpha}{2}\right) \left[\cos\left(\left(\frac{n-1}{2}\right)\alpha\right) + i\sin\left(\left(\frac{n-1}{2}\right)\alpha\right) \right]}{\sin\left(\frac{\alpha}{2}\right)}$$

Taking the imaginary part, we find that -

$$\sum_{n=1}^{\infty} \sin(n\theta) = \frac{\sin\left(\frac{n\theta}{2}\right) \left[\cos(\theta) \sin\left(\left(\frac{n-1}{2}\right)\theta\right) + \sin(\theta) \cos\left(\left(\frac{n-1}{2}\right)\theta\right) \right]}{\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{n\theta}{2}\right) \sin\left(\left(\frac{n+1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$

- 14 In a chemical reaction, substance X decays into substance Y , which in itself goes on to decay.

The rate of decay of X , in grams per hour, is given by 12 times the amount of substance Y , in grams.

The rate of change of Y , in grams per hour, is given by the amount of substance X in grams, minus 7 times the amount of substance Y in grams.

- a) Set up two differential equations for x and y , the amounts, in grams, of the substances X and Y respectively.

[2 marks]

$$\frac{dx}{dt} = -12y \quad (1)$$

$$\frac{dy}{dt} = x - 7y \quad (2)$$

- b) Given that initially $x = 50$ and $y = 0$, solve the coupled system of differential equations you have found in (a).

[8 marks]

From (2)

$$\frac{d^2y}{dt^2} = \frac{dx}{dt} - 7\frac{dy}{dt}$$

$$= -12y - 7\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 12y = 0$$

Hence, the auxiliary equation is

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$$m^2 + 7m + 12 = 0$$

$$(m+3)(m+4) = 0$$

$$\Rightarrow y(t) = Ae^{-3t} + Be^{-4t}$$

$$y(0) = 0 \quad \Rightarrow \quad 0 = A + B \quad (\dagger)$$

Now from (2)

$$\begin{aligned} x &= \frac{dy}{dt} + 7y \\ &= -3Ae^{-3t} - 4Be^{-4t} + 7(Ae^{-3t} + Be^{-4t}) \\ &= 4Ae^{-3t} + 3Be^{-4t} \end{aligned}$$

$$\begin{aligned} f(0) &= 50 \\ \Rightarrow 50 &= -3A - 4B + 7A + 7B \\ \text{so } 50 &= 4A + 3B \quad (\ddagger) \end{aligned}$$

Solving (†) and (‡) gives $A = 50$, $B = -50$

Hence,

$$y(t) = 50e^{-3t} - 50e^{-4t}$$

$$x(t) = 200e^{-3t} - 150e^{-4t}$$

c) Sketch $x(t)$ and $y(t)$ for $0 \leq t \leq 4$.

[2 marks]

