

AQA A-Level Further Maths 2023 Paper

2A

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

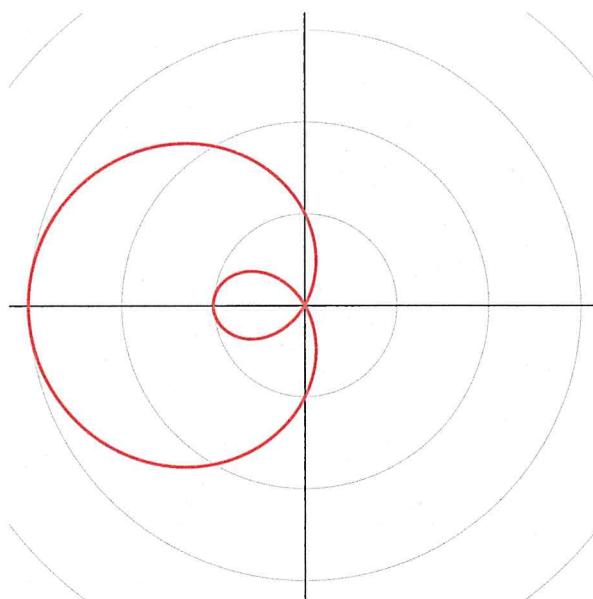
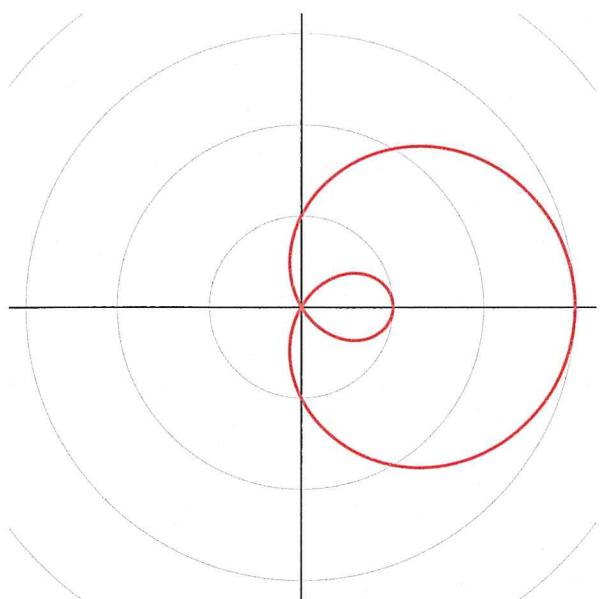
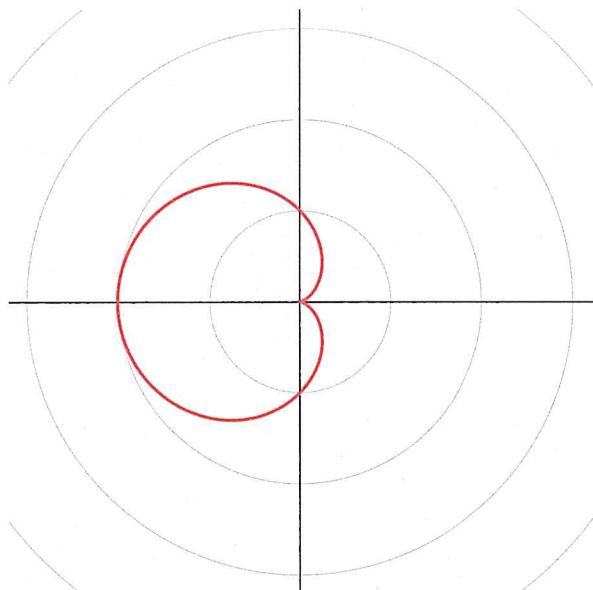
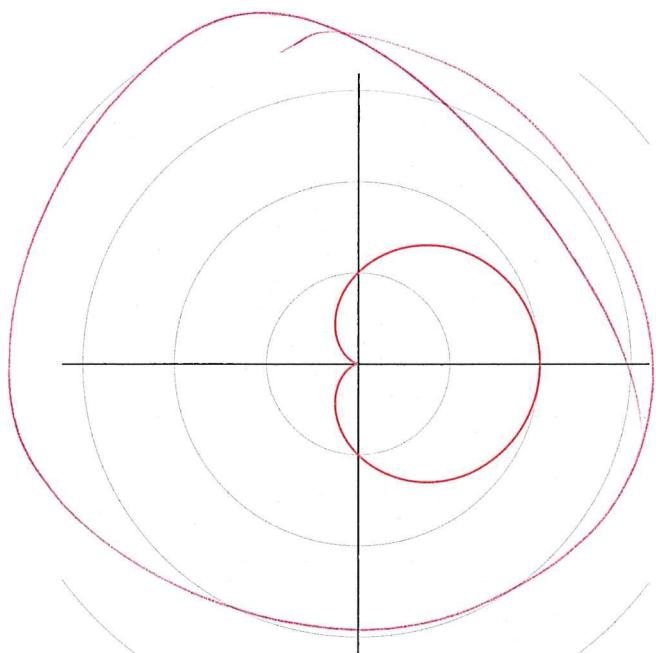
When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100



- 1 Which of the following is a plot of $r = 2 + 2 \cos(\theta)$



[1 mark]

- 2 Let $z_1 = 3 + 2i$ and $z_2 = 1 + 4i$. Then $z_1^* z_2^2$ is

$$29 - 54i$$

$$61 + 6i$$

$$-61 - 6i$$

$$-29 + 54i$$

[1 mark]

$$(1 + 4i)^2 = -15 + 8i$$

$$(3+2i)(1+4i)^2 = -29 + 54i$$

- 3 Which matrix multiplication below will give the solutions to the simultaneous equations $2x + 5y = 16$ and $4x + y = 14$

$$\begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 16 \\ 14 \end{pmatrix}$$

$$\frac{1}{18} \begin{pmatrix} -1 & 5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 16 \\ 14 \end{pmatrix}$$

$$\frac{1}{18} \begin{pmatrix} -1 & 5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

[1 mark]

$$\begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 14 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 16 \\ 14 \end{pmatrix}$$

$$\text{Det} \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} = -18$$

$$= \frac{1}{-18} \begin{pmatrix} -1 & 5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 16 \\ 14 \end{pmatrix}$$

- 4 Show that the curve described by the parametric equations

$$x = 1 + \tan^2(2t), \quad y = 2 \sec(2t)$$

is a parabola.

[3 marks]

$$\begin{aligned}y^2 &= (2 \sec(2t))^2 \\&= 4 \sec^2(2t) \\&= 4(1 + \tan^2(2t)) \\&= 4x\end{aligned}$$

which is the equation of a parabola

5 Simplify $(4\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - 3\mathbf{b})$

[4 marks]

$$(4\underline{\mathbf{a}} + \underline{\mathbf{b}}) \times (2\underline{\mathbf{a}} - 3\underline{\mathbf{b}})$$

$$= 4\underline{\mathbf{a}} \times 2\underline{\mathbf{a}} + 4\underline{\mathbf{a}} \times -3\underline{\mathbf{b}} + \underline{\mathbf{b}} \times 2\underline{\mathbf{a}} + \underline{\mathbf{b}} \times -3\underline{\mathbf{b}}$$

$$= \underbrace{8\underline{\mathbf{a}} \times \underline{\mathbf{a}}}_{=0} - 12\underline{\mathbf{a}} \times \underline{\mathbf{b}} + 2\underline{\mathbf{b}} \times \underline{\mathbf{a}} - \underbrace{3\underline{\mathbf{b}} \times \underline{\mathbf{b}}}_{=0}$$

$$= -12\underline{\mathbf{a}} \times \underline{\mathbf{b}} - 2\underline{\mathbf{a}} \times \underline{\mathbf{b}}$$

$$= -14\underline{\mathbf{a}} \times \underline{\mathbf{b}}$$

b) What would your answer to (a) be if $\underline{\mathbf{b}}$ was parallel to $\underline{\mathbf{a}}$? [1 mark]

0

- 6 Prove by induction $\sum_{r=1}^n r(r!) = (n+1)! - 1$, for all $n \in \mathbb{N}$.

[7 marks]

Step 1: When $n=1$

$$(LHS) = \sum_{r=1}^1 r(r!) = 1$$

~~So~~ $RHS = 2! - 1 = 1$

Step 2: We assume true for $n=k$, i.e.

$$\sum_{r=1}^k r(r!) = (k+1)! - 1$$

Step 3:

$$\sum_{r=1}^{k+1} r(r!) = \sum_{r=1}^k r(r!) + (k+1)((k+1)!) \quad \begin{matrix} \text{To show} \\ \sum_{r=1}^{k+1} r(r!) = (k+2)! - 1 \end{matrix}$$

$$= (k+1)! - 1 + (k+1)((k+1)!) \quad \text{by inductive hypothesis}$$

$$= (1+k+1)((k+1)!) - 1$$

$$= (k+2)(k+1)! - 1$$

$$= (k+2)! - 1$$

Step 4: We have shown the result true for $n=1$, and if true for $n=k$ then it is also true for $n=k+1$. Hence, by the principle of mathematical induction, the result is true for all $n \in \mathbb{N}$

- 7 a) Find the integral to be computer to find the arc length of the curve with equation $y = \ln(1 - 4x^2)$ between $x = a$ and $x = b$.

[3 marks]

$$y = \ln(1 - 4x^2)$$

$$\frac{dy}{dx} = \frac{1}{1 - 4x^2} \times -8x = \frac{-8x}{1 - 4x^2}$$

Hence

$$\text{Arc length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{-8x}{1 - 4x^2}\right)^2} dx$$

$$= \int_a^b \sqrt{1 + \frac{64x^2}{1 - 8x^2 + 16x^4}} dx$$

$$= \int_a^b \sqrt{\frac{1 + 56x^2 + 16x^4}{1 - 8x^2 + 16x^4}} dx$$

- b) Compute the value of this integral in the case that $a = 1$ and $b = 2$, giving your answer to 4 decimal places.

[1 mark]

$$1.90749$$

- 8 Find a closed form expression for $\int \frac{1}{\sqrt{3x^2 + 28x + 43}} dx.$

[5 marks]

$$\begin{aligned}
 \int \frac{1}{\sqrt{3x^2 + 28x + 43}} dx &= \int \frac{1}{\sqrt{3(x+3)^2 + 16}} dx \\
 &= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{(x+3)^2 + \frac{16}{3}}} dx \\
 &= \frac{1}{\sqrt{3}} \operatorname{arsinh} \left(\frac{x+3}{\sqrt{\frac{16}{3}}} \right) + C \\
 &= \frac{1}{\sqrt{3}} \operatorname{arsinh} \left(\frac{\sqrt{3}(x+3)}{4} \right) + C
 \end{aligned}$$

- 9 Find the Maclaurin series expansion of $y = \sqrt[3]{4x+3}$ up to the term in x^3 . Show all your reasoning.

[6 marks]

$$\begin{aligned}
 y &= (4x+3)^{\frac{1}{3}} & y(0) &= 3^{\frac{1}{3}} \\
 y'(x) &= \frac{1}{3} (4x+3)^{-\frac{2}{3}} \times 4 & y'(0) &= \frac{4}{3 \times 3^{\frac{2}{3}}} \\
 y''(x) &= -\frac{2}{9} (4x+3)^{-\frac{5}{3}} \times 4 \times 4 & y''(0) &= -\frac{32}{27 \times 3^{\frac{2}{3}}} \\
 y'''(x) &= \frac{640}{27} (4x+3)^{-\frac{8}{3}} & y'''(0) &= \frac{640}{243 \times 3^{\frac{2}{3}}}
 \end{aligned}$$

Now, the Maclaurin Series is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$$

$$= 3^{\frac{1}{3}} + \frac{4\sqrt[3]{3}}{9}x - \frac{16\sqrt[3]{3}}{81}x^2 + \frac{320\sqrt[3]{3}}{2187}x^3$$

- 10 a) Show that for $n \geq 2$, where $I_n = \int_0^a x^n \cosh(x) dx$, that the reduction formula holds.

$$I_n = n(n-1)I_{n-2} + a^n \sinh(a) - na^{n-1} \cosh(a)$$

[7 marks]

Consider

$$I_n = \int_0^a x^n \cosh(x) dx \quad \text{Let } u = x^n \quad \frac{du}{dx} = nx^{n-1}, \quad v = \sinh(x)$$

$$\begin{aligned} I_n &= \left[x^n \sinh(x) \right]_0^a - \int_0^a nx^{n-1} \sinh(x) dx \\ &= a^n \sinh(a) - n \int_0^a x^{n-1} \sinh(x) dx \quad \text{Let } u = x^{n-1} \quad \frac{du}{dx} = \frac{n-1}{n} x^{n-2} \\ &= a^n \sinh(a) - n \left[\left[x^{n-1} \cosh(x) \right]_0^a - (n-1) \int_0^a x^{n-2} \cosh(x) dx \right] \\ &= a^n \sinh(a) - na^{n-1} \cosh(a) + n(n-1)I_{n-2} \end{aligned}$$

So

$$I_n = n(n-1)I_{n-2} + a^n \sinh(a) - na^{n-1} \cosh(a)$$

- b) Using (a) show that $\int_0^1 x^4 \cosh(x) dx = \frac{9}{2}e - \frac{65}{2e}$

[5 marks]

$$I_0 = \int_0^1 \cosh^4(x) dx = \left[\sinh(x) \right]_0^1 \\ = \sinh(1)$$

$$I_2 = 2 \times 1 \times I_0 + \sinh(1) - 2 \cosh(1) \\ = 2 \sinh(1) + \sinh(1) - 2 \cosh(1) \\ = 3 \sinh(1) - 2 \cosh(1)$$

$$\begin{aligned} I_4 &= 4 \times 3 \times I_2 + \sinh(1) - 4 \cosh(1) \\ &= 12(3 \sinh(1) - 2 \cosh(1)) + \sinh(1) - 4 \cosh(1) \\ &= 37 \sinh(1) - 28 \cosh(1) \\ &= 37 \left(\frac{e^1 - e^{-1}}{2} \right) - 28 \left(\frac{e^1 + e^{-1}}{2} \right) \\ &= \frac{37e^1 - 37e^{-1} - 28e^1 - 28e^{-1}}{2} \\ &= \frac{9e^1 - 65e^{-1}}{2} \\ &= \frac{9}{2}e^1 - \frac{65}{2e} \end{aligned}$$

11 a) The solutions, α , β and γ to the equation

$$27z^3 + bz^2 + 152z - 64 = 0$$

form a geometric progression. Solve this equation finding all three roots.

[5 marks]

Let $\frac{\alpha}{r}$, a and ar be expressions for α , β , γ respectively.

Then, Vieta's formulae tell us,

$$\frac{\alpha}{r} + a + ar = -\frac{b}{27} \quad (1)$$

$$\frac{\alpha}{r}a + \frac{\alpha}{r}ar + aar = \frac{152}{27} \quad (2)$$

and $\frac{\alpha}{r}a + ar = \frac{64}{27} \quad (3)$

$$(3) \Rightarrow a^3 = \frac{64}{27} \Rightarrow a = \frac{4}{3}$$

Using (2) $\frac{a^2}{r} + a^2 + ar = \frac{152}{27}$

$$\Rightarrow a^2 \left(\frac{1}{r} + 1 + r \right) = \frac{152}{27}$$

$$\Rightarrow \frac{1}{r} + 1 + r = \frac{19}{6}$$

$$\Rightarrow 1 + r^2 = \frac{13}{6}r$$

$$\Rightarrow r^2 - \frac{13}{6}r + 1 = 0$$

$$\Rightarrow r = \frac{3}{2} \text{ or } \frac{2}{3}$$

Hence the roots are

$$\frac{8}{9}, \frac{4}{3} \text{ and } 2$$

- b) Find the value of b .

[2 marks]

Using ①

$$\frac{8}{9} + \frac{b}{3} + 2 = -\frac{b}{27}$$

$$\Rightarrow b = -114$$

- c) Find a cubic equation with roots $\alpha - 1$, $\beta - 1$ and $\gamma - 1$.

[2 marks]

$$\text{Let } w = x - 1 \Rightarrow x = w + 1$$

Then

$$27(w+1)^3 - 114(w+1)^2 + 152(w+1) - 64 = 0$$

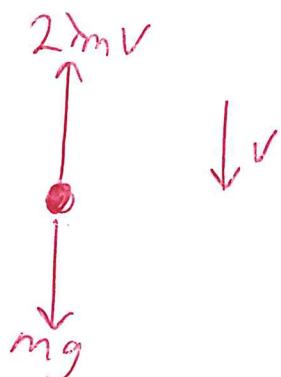
$$\Rightarrow 27w^3 - 33w^2 + 5w + 1 = 0 \text{ has solutions } \alpha^{-1}, \beta^{-1} \text{ and } \gamma^{-1}$$

- 12 a) A ball of mass m falls vertically downwards (under gravity) through a vat of liquid.

At time t the ball has speed v and it experiences a resistive force of magnitude $2\lambda mv$.

Show that

$$\frac{dv}{dt} = g - 2\lambda v$$



[3 marks]

Apply $F=ma$

$$ma = mg - 2\lambda mV$$

$$\Rightarrow m \frac{dv}{dt} = mg - 2\lambda mV$$

$$\Rightarrow \frac{dv}{dt} = g - 2\lambda V$$

- b) At time $t = 0$, $v = u$. Find an expression for $v(t)$.

$$\begin{aligned} \frac{dv}{dt} &= g - 2\lambda v & [\text{5 marks}] \\ \Rightarrow \int \frac{1}{g - 2\lambda v} dt &= \int 1 dt \\ \Rightarrow -\frac{1}{2\lambda} \ln |g - 2\lambda v| &= t + C \\ \Rightarrow \ln |g - 2\lambda v| &= -2\lambda t + 2\lambda C \\ \text{so } g - 2\lambda v &= A e^{-2\lambda t} \\ \Rightarrow v(t) &= \frac{g - A e^{-2\lambda t}}{2\lambda} \end{aligned}$$

$$\text{When } t=0, v=u, \text{ so } u = \frac{g - A}{2\lambda} \Rightarrow A = g - 2\lambda u$$

Hence

$$v(t) = \frac{g - (g - 2\lambda u) e^{-2\lambda t}}{2\lambda}$$

- 13 Three points in three dimensional space have coordinates $A(-1, -2, 1)$, $B(2, 4, 2)$ and $C(1, 3, 0)$.

Find the shortest distance from the line passing through AB and the point C .

[8 marks]

$$\vec{AB} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$$

Hence, equation of the line through A and B is

$$L = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$$

Let P be on the line, then

$$\begin{aligned} \vec{CP} &= \vec{OP} - \vec{OC} \\ &= \begin{pmatrix} -1+3\lambda \\ -2+6\lambda \\ 1+\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2+3\lambda \\ -5+6\lambda \\ 1+\lambda \end{pmatrix} \end{aligned}$$

$$\text{So } \vec{CP} \cdot \vec{AB} = 0 \quad \text{(for them to be } \perp \text{ to get shortest distance)}$$

$$\Rightarrow \begin{pmatrix} -2+3\lambda \\ -5+6\lambda \\ 1+\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$$

$$\Rightarrow -6 + 9\lambda - 30 + 36\lambda + 1 + \lambda = 0$$

$$\Rightarrow 46\lambda = 35$$

$$\lambda = \frac{35}{46}$$

Hence

$$CP = \begin{pmatrix} 13/46 \\ -10/23 \\ 81/46 \end{pmatrix}$$

And so

$$|CP| = \sqrt{\left(\frac{13}{46}\right)^2 + \left(\frac{-10}{23}\right)^2 + \left(\frac{81}{46}\right)^2}$$

$$= \sqrt{\frac{155}{46}}$$

$$\approx 1.836$$

14 a) Show that the series

$$1 + \frac{1}{4}e^{i\theta} + \frac{1}{16}e^{2i\theta} + \frac{1}{64}e^{3i\theta} + \dots$$

converges and find an expression for its sum to infinity.

[3 marks]

Geometric series

$$|r| = \left| \frac{1}{4} e^{i\theta} \right|$$

$$= \frac{1}{4}$$

< 1 , hence it converges.

Using $S_\infty = \frac{a}{1-r}$ with $a=1$, $r = \frac{1}{4} e^{i\theta}$

$$S_\infty = \frac{1}{1 - \frac{1}{4} e^{i\theta}}$$

$$= \frac{4}{4 - e^{i\theta}}$$

b) Hence, find $\sum_{k=0}^{\infty} \frac{1}{4^k} \cos(k\theta)$

[7 marks]

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{1}{4^k} \cos(k\theta) &= \operatorname{Re} \left(1 + \frac{1}{4} e^{i\theta} + \frac{1}{16} e^{2i\theta} + \frac{1}{64} e^{3i\theta} + \dots \right) \\
 &= \operatorname{Re} \left(\frac{4}{4 - e^{i\theta}} \right) \\
 &= \operatorname{Re} \left(\frac{4}{4 - e^{i\theta}} \times \frac{4 - e^{-i\theta}}{4 - e^{-i\theta}} \right) \\
 &= \operatorname{Re} \left(\frac{16 - 4e^{-i\theta}}{16 - 4e^{-i\theta} - 4e^{i\theta} + 1} \right) \\
 &= \operatorname{Re} \left(\frac{16 - 4e^{-i\theta}}{17 - 8\cos\theta} \right) \\
 &= \frac{16 - 4\cos\theta}{17 - 8\cos\theta}
 \end{aligned}$$

- 15 A curve C_1 has equation $y = f(x)$ where $f(x) = \frac{x^2 + x + 1}{x^2 + 4x + 1}$.

The line $y = k$ intersects the curve C_1 .

- a) Show that $3(4k^2 - 1) \geq 0$

[4 marks]

$$\frac{f(x)}{x^2 + 4x + 1} = \frac{x^2 + x + 1}{(x+3)(x+1)}$$

Consider,

$$k = \frac{x^2 + x + 1}{(x+3)(x+1)} \quad \frac{x^2 + x + 1}{x^2 + 4x + 1}$$

$$\Rightarrow k(x^2 + 4x + 1) = x^2 + x + 1$$

$$\Rightarrow (k-1)x^2 + (4k-1)x + (k-1) = 0$$

As the line intersects the curve, the discriminant ≥ 0

$$(4k-1)^2 - 4(k-1)(k-1) \geq 0$$

$$\Rightarrow 12k^2 - 3 \geq 0$$

$$\Rightarrow 3(4k^2 - 1) \geq 0$$

- b) Hence, find the coordinates of the stationary points of C_1 .

[4 marks]

At the stationary points

$$3(4k^2 - 1) = 0$$

$$\text{So } k^2 = \frac{1}{4}$$

$$k = \pm \frac{1}{2}$$

$$\text{When } k = \frac{1}{2}, \quad -\frac{x^2}{2} + x - \frac{1}{2} = 0$$

$\Rightarrow x = 1$ so $(1, \frac{1}{2})$ is a stationary point

$$\text{When } k = -\frac{1}{2}$$

$$\frac{-3x^2 - 3x - 3}{2} = 0$$

$\Rightarrow x = -1$ so $(-1, -\frac{1}{2})$ is a stationary point

- c) What are the asymptotes of C_1 ?

[3 marks]

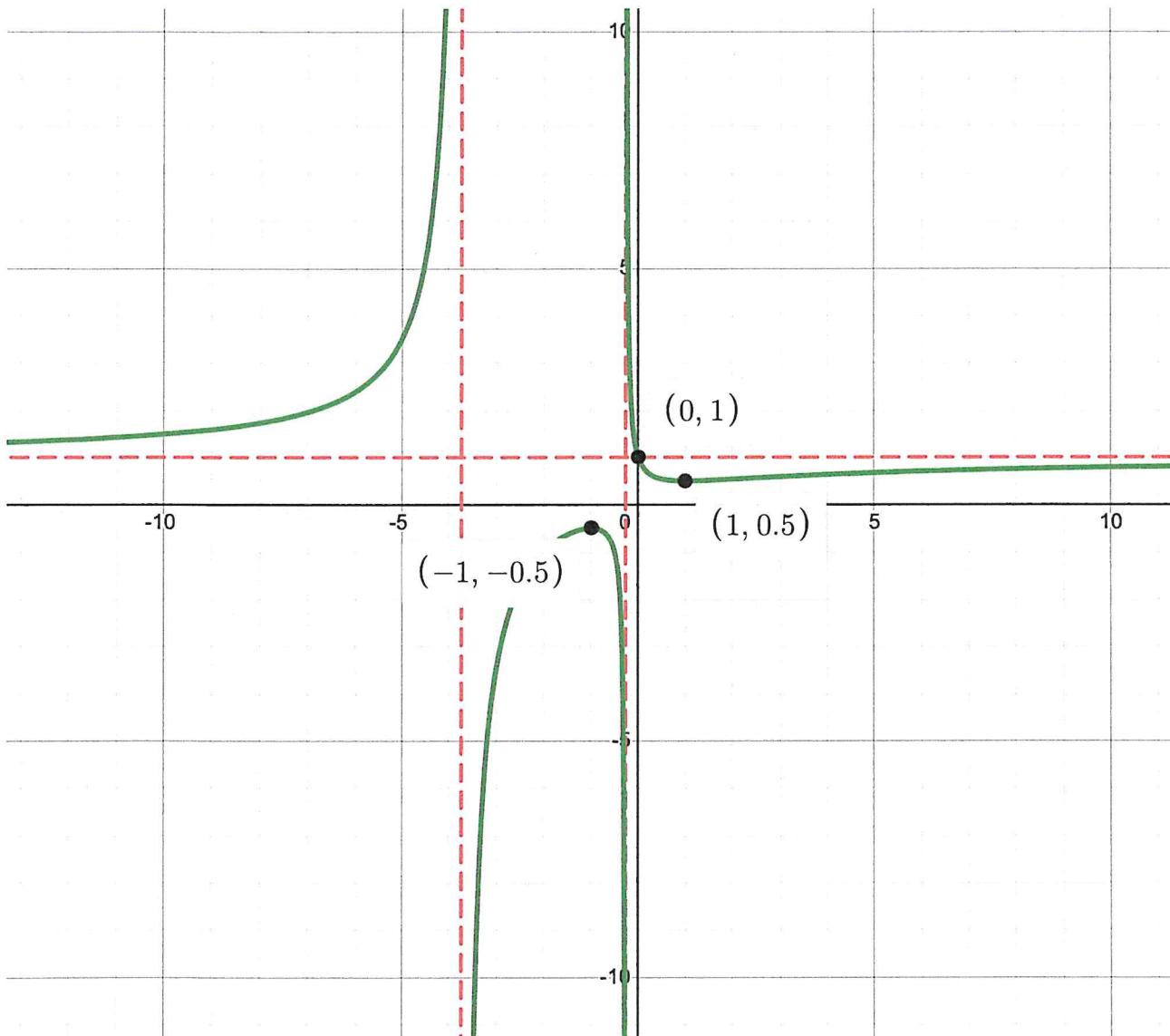
$$x = -2 - \sqrt{3}$$

$$x = \sqrt{3} - 2$$

$$y = 1$$

d) Sketch C_1 .

[3 marks]



The C_2 has equation $y = \frac{1}{f(x)}$.

- e) What can we immediately say about the coordinates of the stationary points of C_2 ?

[1 mark]

They will have the same x-coordinates as the stationary points of $f(x)$.

- f) Hence, state the stationary points of C_2 .

[2 marks]

$$(-1, -2)$$

$$\text{and } (1, 2)$$

- g) Show that C_2 has no vertical asymptotes.

[2 marks]

Discriminant of $x^2 + 4x + 1$ is

$$4^2 - 4 \times 1 \times 1 = 16 - 4 = 12 > 0$$

And so the denominator of $\frac{1}{f(x)}$ is never zero, hence no vertical asymptotes.

h) Sketch C_2 **[2 marks]**