

AQA A-Level Further Maths 2023 Paper 1

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100

SOLUTION)



- 1 In the triangle $ABC \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$.

The area of the triangle ABC is

$2\sqrt{6}$

$\sqrt{6}$

8

4

[1 mark]

$$\begin{aligned} \text{Area } g\Delta ABC &= \frac{1}{2} |\vec{AB} \times \vec{AC}| \\ &= \frac{1}{2} (2^2 + 2^2 + 4^2)^{1/2} \\ &= \frac{1}{2} \times 2\sqrt{6} \\ &= \sqrt{6} \end{aligned}$$

- 2 The mean value of the function $f(x) = x^2 + 2$ over the interval $[1, 5]$ is

$\frac{37}{3}$

$\frac{148}{3}$

$\frac{74}{9}$

37

[1 mark]

$$\begin{aligned} \frac{1}{5-1} \int_1^5 x^2 + 2 dx &= \frac{1}{4} \times \frac{148}{3} \\ &= \frac{37}{3} \end{aligned}$$

- 3 The matrix A has an determinant of 2 and the matrix B has a determinant of 3. What is the determinant of AB^2 ?

2

18

3

12

[1 mark]

$$\det(AB^2) = 2 \times 3^2 \\ = 18$$

- 4 Show that the vectors $\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -7 \\ -8 \end{pmatrix}$ are perpendicular.

[3 marks]

$$\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -7 \\ -8 \end{pmatrix} = 1 \cdot 3 + 5 \cdot (-7) + (-4) \cdot (-8) \\ = 0$$

As the scalar product is zero the vectors are perpendicular

- 5 a) Phil is attempting to prove the statement
 $P(n) : 2^n < n!$, $\forall n \geq 1$. Explain why he will fail to do so.

[2 marks]

$$2^2 = 4$$

$$2! = 2$$

Since $2^2 > 2!$ the statement $P(n)$ is not true for all natural numbers.

- b) State a corrected version of this statement and prove it by induction.

[6 marks]

Let $P(1)$ be the statement

$$= P(n) : 2^n < n!, \forall n \geq 4, n \in \mathbb{N}^*$$

Base Case: When $n=4$,

$$2^4 = 16$$

$$4! = 24$$

Hence $2^4 < 4!$ and $P(4)$ is true

Step 2: We assume $P(k)$ is true, i.e. $2^k < k!$, where $k \geq 4$

Step 3: Now for $n = k+1$,

$$\begin{aligned}
 2^{k+1} &= 2 \times 2^k \\
 &< 2k! && \text{by inductive hypothesis} \\
 &< (k+1)k! && \text{since } k \geq 4 \\
 &= (k+1)!
 \end{aligned}$$

Step 4: Since $P(k)$ is true, and if true for $n=k$, we have shown the truth of $P(k+1)$, by the principle of mathematical induction has been shown to be true for all $n \geq 4$, $n \in \mathbb{N}$

- 6 a) Show that $\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.

[4 marks]

$$y = \arcsin(x) \Rightarrow \sin(y) = x$$

Differentiating w.r.t. x,

$$\cos(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

since $x = \sin(y)$
and $\sin^2(y) + \cos^2(y) = 1$

- b) Hence, find the derivative of $y = \sin^{-1}(x^2 + 3x)$

[2 marks]

Let $u = x^2 + 3x$, then by the chain rule

$$\frac{dy}{dx} = \frac{d}{du} (\sin^{-1}(u)) \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-(x^2+3x)^2}} \times 2x+3$$

$$= \frac{2x+3}{\sqrt{1-(x^2+3x)^2}}$$

- 7 a) Show that the Maclaurin series of $\ln(1 + x)$ is given by

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Let $f(x) = \ln(1+x)$

[4 marks]

Then,

$$f(0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{1+x}$$

$$f'(0) = 1$$

$$f''(x) = -(1+x)^{-2}$$

$$f''(0) = -1$$

$$f'''(x) = 2(1+x)^{-3}$$

$$f'''(0) = 2$$

$$f^{(iv)}(x) = -6(1+x)^{-4}$$

$$f^{(iv)}(0) = -6$$

Hence,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(iv)}(0)}{4!}x^4 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

b) Hence, find the Maclaurin series for $\ln(3 + x)$.

[3 marks]

$$\ln(3 + x) = \ln\left(3\left(1 + \frac{x}{3}\right)\right)$$

$$= \ln(3) + \ln\left(1 + \frac{x}{3}\right)$$

$$= \ln(3) + \left(\frac{x}{3}\right) - \frac{1}{2}\left(\frac{x}{3}\right)^2 + \frac{1}{3}\left(\frac{x}{3}\right)^3 - \frac{1}{4}\left(\frac{x}{3}\right)^4 + \dots$$

$$= \ln(3) + \frac{x}{3} - \frac{x^2}{18} + \frac{x^3}{81} - \frac{x^4}{324} + \dots$$

8 a) Prove the identity

$$\cosh^2(x) - \sinh^2(x) \equiv 1$$

[2 marks]

$$\begin{aligned}
 \text{LHS: } \cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
 &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right) \\
 &= \frac{4}{4} \\
 &= 1
 \end{aligned}$$

b) Solve $3 \sinh^2(x) - 11 \cosh(x) + 9 = 0$, leaving your answers in exact form.

[6 marks]

$$\begin{aligned}
 3 \sinh^2(x) - 11 \cosh(x) + 9 &= 0 \\
 \Rightarrow 3(\cosh^2(x) - 1) - 11 \cosh(x) + 9 &= 0 \\
 \Rightarrow 3 \cosh^2(x) - 11 \cosh(x) + 6 &= 0 \\
 (3 \cosh(x) - 2)(\cosh(x) - 3) &= 0
 \end{aligned}$$

Mence $3\cosh(x) - 2 = 0 \Rightarrow \cosh(x) = \frac{2}{3}$ which has no real solutions

and

$$\cosh(x) - 3 = 0 \Rightarrow \cosh(x) = 3$$

so

$$\begin{aligned} x &= \operatorname{arccosh}(3) \\ &= \ln(3 + \sqrt{8}) \end{aligned}$$

9 a) Show that $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$

[2 marks]

$$\begin{aligned} \frac{1}{r^2} - \frac{1}{(r+1)^2} &= \frac{(r+1)^2 - r^2}{r^2(r+1)^2} \\ &= \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2} \\ &= \frac{2r+1}{r^2(r+1)^2} \end{aligned}$$

b) Hence, find the sum of the first n terms of the series

$$\frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} + \frac{11}{900} + \dots$$

[4 marks]

When :

$$r=1 \quad 1 - \frac{1}{2^2}$$

$$r=2 \quad \frac{1}{2^2} - \frac{1}{3^2}$$

$$r=3 \quad \frac{1}{3^2} - \frac{1}{4^2}$$

.

.

.

$$r=n+1 \quad \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}$$

$$r=n \quad \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

Hence,

$$\underbrace{\frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \dots + \frac{2n+1}{n^2(n+1)^2}}_{n \text{ terms}} = 1 - \frac{1}{(n+1)^2}$$

$$= \frac{(n+1)^2 - 1}{(n+1)^2}$$

$$= \frac{n^2 + 2n}{(n+1)^2}$$

- 10 a) Find, without using calculus the turning points of the function

$$y = \frac{x^2 - x - 6}{x^2 - x - 12}$$

[6 marks]

Let $y = k$

$$\begin{aligned} k &= \frac{x^2 - x - 6}{(x+3)(x-4)} \\ &= \frac{x^2 - x - 6}{x^2 - x - 12} \end{aligned}$$

$$\Rightarrow k(x^2 - x - 12) = x^2 - x - 6$$

$$\Rightarrow (k-1)x^2 + (1-k)x + (6-12k) = 0$$

Turning point when discriminant is 0.

So

$$(1-k)^2 - 4(k-1)(6-12k) = 0$$

$$\Rightarrow 1 - 2k + k^2 - 4(6k - 12k^2 - 6 + 12k) = 0$$

$$\Rightarrow 1 - 2k + k^2 + 48k^2 - 72k + 24 = 0$$

$$\Rightarrow 49k^2 - 74k + 25 = 0$$

So

$$k = 1 \text{ or } k = \frac{25}{49}$$

When $k = 1$ it is not a turning point.

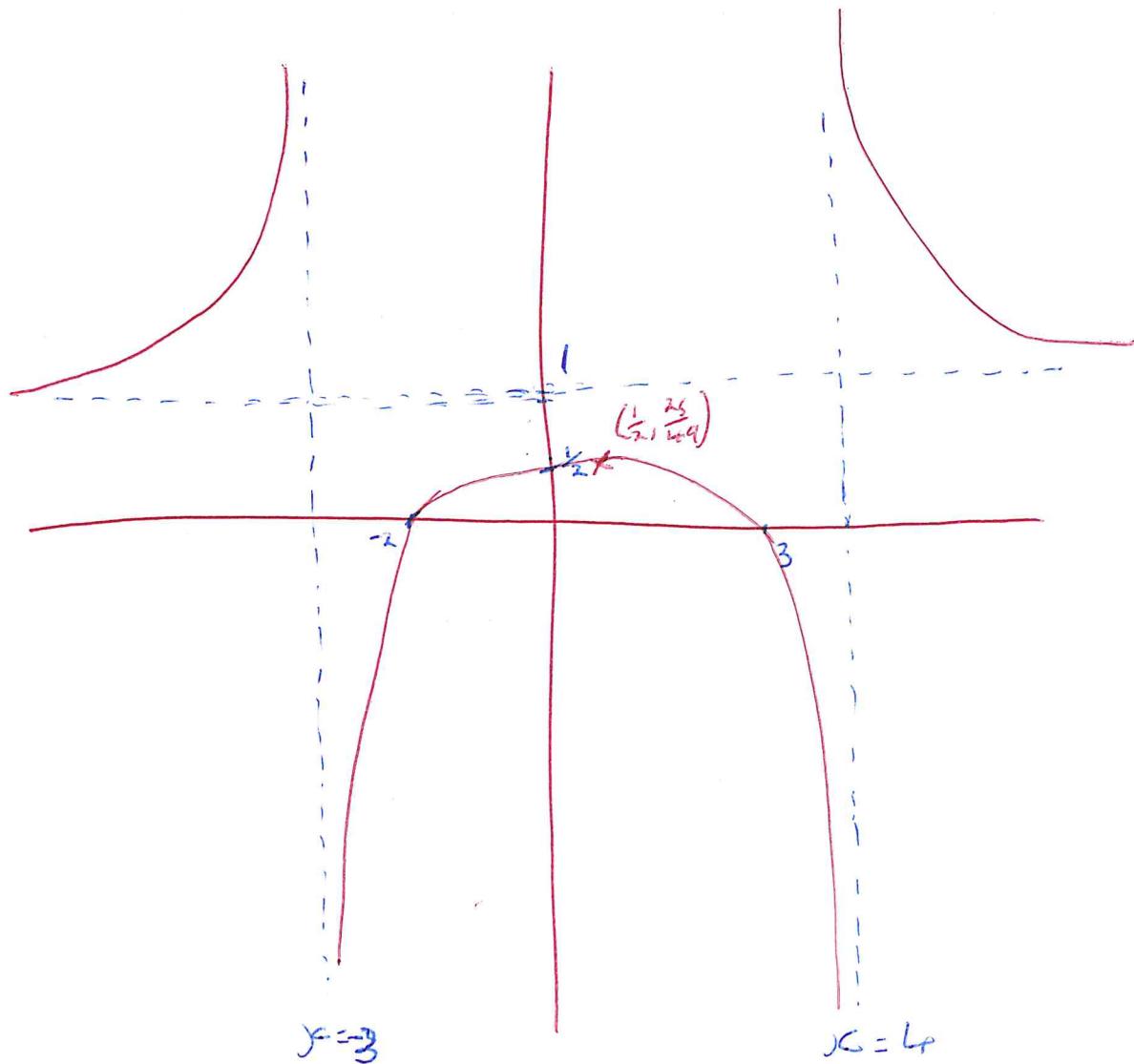
$$\text{When } k = \frac{25}{49} \quad \begin{aligned} 24x^2 - 24x - 6 &= 0 \\ \Rightarrow 6(2x-1)^2 &= 0 \quad \Rightarrow x = \frac{1}{2} \end{aligned}$$

So turning point is $(\frac{1}{2}, \frac{25}{49})$

$$\begin{aligned} a &= k-1 \\ b &= 1-k \\ c &= 6-12k \end{aligned}$$

- b) Sketch $y = \frac{x^2 - x - 6}{x^2 - x - 12}$ labelling all important features of the graph.

[4 marks]

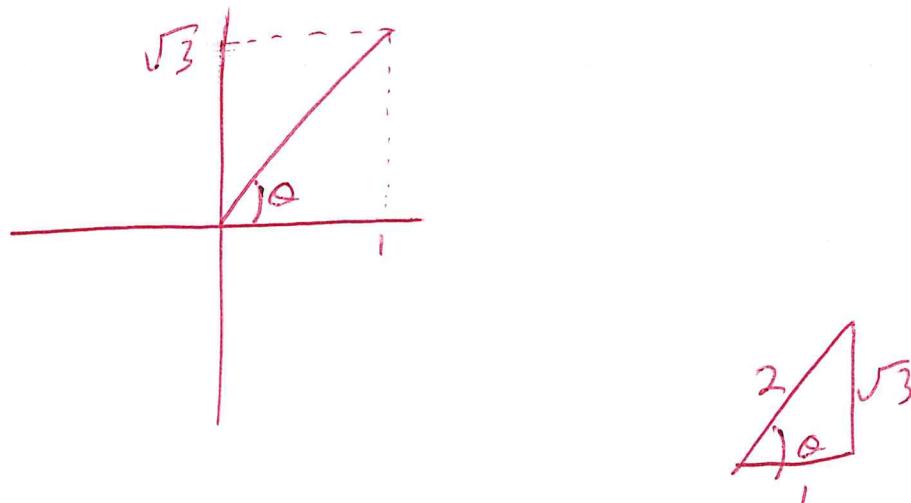


Asymptotes at $x = -3$ and $x = 4$

Asymptote at $y = 1$

- 11 a) Show the complex number $z = 1 + \sqrt{3}i$ on an Argand diagram.

[1 mark]



- b) Show that z^{-3} is a purely real number.

[4 marks]

$$|z| = \sqrt{3+1} = \sqrt{4} = 2$$

$$\arg(z) = \frac{\pi}{3}$$

Hence,

$$\begin{aligned} z^{-3} &= \left(2 \left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)^{-3} \\ &= 2^{-3} \left(\cos\left(-\frac{3\pi}{3}\right) + i\sin\left(-\frac{3\pi}{3}\right)\right) \\ &= \frac{1}{8} \left(\cos(-\pi) + i\sin(-\pi)\right) \\ &= -\frac{1}{8} \end{aligned}$$

which is purely real.

c) Find rational numbers A, B and C such that

$$\int \cos^4(\theta) = A \sin(4\theta) + B \sin(2\theta) + C\theta + \text{constant of integration}$$

[7 marks]

$$\begin{aligned} (2 \cos(\theta))^4 &= 16 \cos^4(\theta) \\ &= (e^{i\theta} + e^{-i\theta})^4 \\ &= (e^{i\theta})^4 + 4(e^{i\theta})(e^{-i\theta})^3 + 6(e^{i\theta})^2(e^{-i\theta})^2 \\ &\quad + 4(e^{i\theta})(e^{-i\theta})^3 + (e^{-i\theta})^4 \\ &= e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta} \\ &= (e^{4i\theta} + e^{-4i\theta}) + 4(e^{2i\theta} + e^{-2i\theta}) + 6 \\ &= 2\cos(4\theta) + 8\cos(2\theta) + 6 \end{aligned}$$

Hence, $\cos^4\theta = \frac{1}{16} \cos(4\theta) + \frac{8}{16} \cos(2\theta) + \frac{6}{16}$

Hence

$$\begin{aligned} \int \cos^4(\theta) d\theta &= \frac{1}{8} \int \cos(4\theta) d\theta + \frac{1}{2} \int \cos(2\theta) d\theta + \frac{3}{8} \int d\theta \\ &= \frac{1}{8} \left[\frac{1}{4} \sin(4\theta) \right] + \frac{1}{2} \left[\frac{1}{2} \sin(2\theta) \right] + \frac{3}{8} \theta \\ &= \frac{1}{32} \sin(4\theta) + \frac{1}{4} \sin(2\theta) + \frac{3}{8} \theta \end{aligned}$$

- 12 a) Find all invariant lines of the form $y = mx + c$ for the transformation represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$$

[7 marks]

Consider

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix} \begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} -\frac{12}{13}x + \frac{5}{13}(mx+c) \\ \frac{5}{13}x + \frac{12}{13}(mx+c) \end{pmatrix}$$

but x' and y' satisfy $y' = mx' + c$

$$\frac{5}{13}x + \frac{12}{13}(mx+c) = m \left(-\frac{12}{13}x + \frac{5}{13}(mx+c) \right) + c$$

$$\Rightarrow \frac{5}{13}x + \frac{12}{13}mx + \frac{12}{13}c = -\frac{12}{13}mx + \frac{5}{13}m^2x + \frac{5}{13}mc + \cancel{13c} + c$$

$$\Rightarrow \cancel{\frac{5}{13}m^2x} + \cancel{\left(-\frac{24}{13}m - \frac{5}{13} \right)} + c \left(1 + \cancel{\frac{7}{13}} \right)$$

$$5x + 12mx + 12 = -12mx + 5m^2x + 5mc + 13c$$

$$\Rightarrow 5m^2x - 24mx - 5x + 5mc + c = 0$$

$$\text{So } x(5m^2 - 24m - 5) + c(5m + 1) = 0$$

$$x(5m + 1)(m - 5) + c(5m + 1) = 0$$

$$\text{Hence } m = -\frac{1}{5} \text{ or } 5$$

When $m = -\frac{1}{5}$, c can be anything so invariant

lines of the form $y = -\frac{1}{5}x + c$

When $m = 5$, $c = 0$, so invariant line $y = 5x$

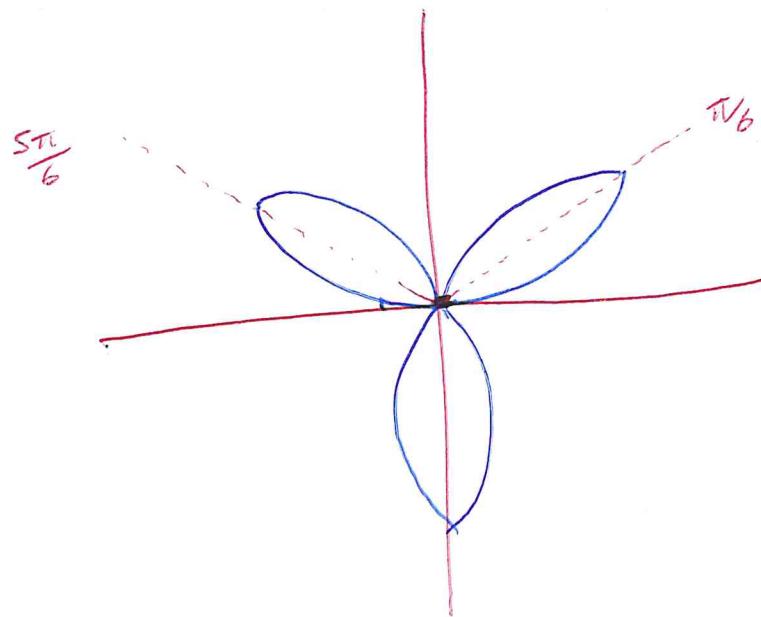
- b) Using part (a), deduce the transformation represented by \mathbf{M} .

[2 marks]

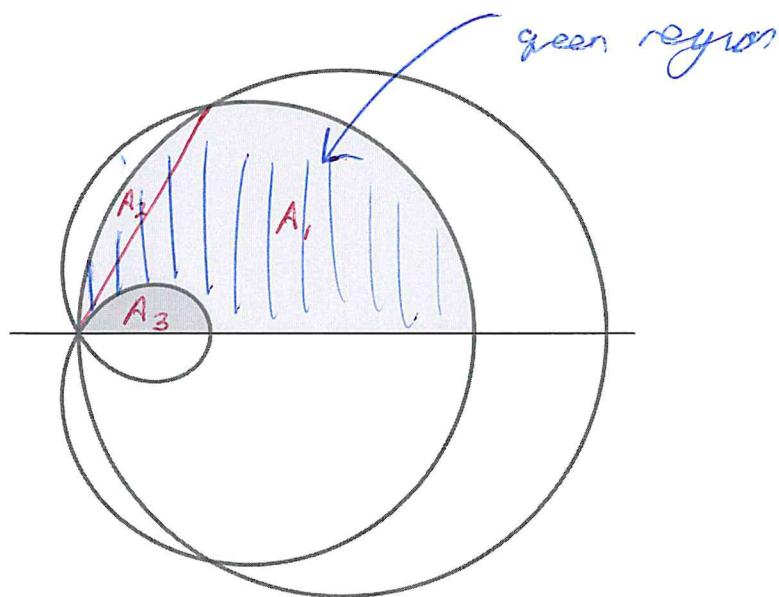
Reflection in the line $y = 5x$

- 13 a) Sketch the curve given by $r = \sin(3\theta)$, $0 \leq \theta \leq 2\pi$.

[3 marks]



- b) The polar curves $r = 4 \cos(\theta)$ and $r = 1 + 2 \cos(\theta)$ are shown below.



Find the area of the green region shaded above.

[9 marks]

$r = 4\cos\theta$ and $r = 1 + 2\cos\theta$ intersect at $\frac{\pi}{3}$

$$\text{Total Area} = A_1 + A_2 - A_3$$

$$A_1 = \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 + 2\cos\theta)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 + 4\cos\theta + 2 + 2\cos 2\theta) d\theta$$

$$= \frac{1}{2} \left[\theta + 4\sin\theta + 2\theta + \sin 2\theta \right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2} \left[\pi + \frac{4\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right]$$

$$= \frac{\pi}{2} + \frac{5\sqrt{3}}{4}$$

$$A_2 = \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (4\cos\theta)^2 d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 16\cos^2\theta d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 16 \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 8 + 8\cos(2\theta) d\theta$$

$$= \frac{1}{2} \left[8\theta + L \sin 2\theta \right]_{\pi/3}^{\pi/2}$$

$$= \frac{L}{2} \left[4\pi + 0 - \frac{8\pi}{3} - \frac{4\sqrt{3}}{2} \right]$$

$$= \frac{L}{2} \left[\frac{4\pi}{3} - 2\sqrt{3} \right]$$

$$= \frac{2\pi}{3} - \sqrt{3}$$

$$A_3 = \frac{1}{2} \int_{\pi}^{\frac{4\pi}{3}} (1+2\cos\theta)^2$$

$$= \frac{1}{2} \left[3\theta + L \sin(\theta) + \sin(2\theta) \right]_{\pi}^{\frac{4\pi}{3}}$$

$$= \frac{1}{2} \left[4\pi - \frac{4\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 3\pi + 0 + 0 \right]$$

$$= \frac{1}{2} \left[\pi - \frac{3\sqrt{3}}{2} \right]$$

$$= \frac{\pi}{2} - \frac{3\sqrt{3}}{4}$$

Hence, total area is

$$A_1 + A_2 - A_3 = \frac{\pi}{2} + \frac{3\sqrt{3}}{4} + \frac{2\pi}{3} - \sqrt{3} - \frac{\pi}{2} + \frac{3\sqrt{3}}{4}$$

$$= \frac{2\pi}{3} + \sqrt{3}$$

- 14 a) Show that for $I_n = \int \frac{1}{(a^2 - x^2)^n} dx$ the reduction formula below is true.

$$I_n = \frac{x}{2a^2(n-1)(a^2 - x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}$$

[7 marks]

$$I_n = \int (a^2 - x^2)^{-n} dx \quad \text{Let } u = (a^2 - x^2)^{-n} \quad \underline{du} =$$

$$\frac{du}{dx} = 2x n (a^2 - x^2)^{n-1} \quad v = x$$

Using IBP $\int u v dx = uv - \int v du$

Hence,

$$I_n = \frac{x}{(a^2 - x^2)^n} - 2n \int \frac{x^2}{(a^2 - x^2)^{n+1}} dx$$

$$= \frac{x}{(a^2 - x^2)^n} + 2n \left[\int \frac{(a^2 - x^2)}{(a^2 - x^2)^{n+1}} dx - \int \frac{a^2}{(a^2 - x^2)^{n+1}} dx \right]$$

$$= \frac{x}{(a^2 - x^2)^n} + 2n I_n - 2n a^2 I_{n+1}$$

$$\Rightarrow 2n a^2 I_{n+1} = (2n-1) I_n + \frac{x}{2n a^2 (a^2 - x^2)^n}$$

$$\Rightarrow I_{n+1} = \frac{(2n-1)}{2n a^2} I_n + \frac{x}{2n a^2 (a^2 - x^2)^n}$$

Change variables

$$n+1 \mapsto n \quad \text{or} \quad n \mapsto n-1$$

So

$$\begin{aligned} I_n &= \frac{(2(n-1) - 1)}{2(n-1)a^2} I_{n-1} + \frac{x}{2(n-1)a^2(a^2-x^2)^{n-1}} \\ &= \frac{x}{2(n-1)a^2(a^2-x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1} \end{aligned}$$

as required

b) Hence, find a closed form for $I_2 = \int \frac{1}{(a^2 - x^2)^2} dx$

Using (a)

[2 marks]

$$\begin{aligned}
 I_2 &= \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{2a^2} I, \\
 &= \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{2a^2} \int \frac{1}{a^2 - x^2} dx \\
 &= \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{2a^2} \cdot \frac{1}{a} \operatorname{artanh}\left(\frac{x}{a}\right) \\
 &= \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{2a^3} \operatorname{artanh}\left(\frac{x}{a}\right)
 \end{aligned}$$

c) Show that

$$\operatorname{artanh}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

[4 marks]

Let $y = \operatorname{artanh}(x)$

$$\Rightarrow \tanh(y) = x$$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$= \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$\Rightarrow xe^{2y} + x = e^{2y} - 1$$

$$\Rightarrow 1 + x = e^{2y}(1 - x)$$

$$\Rightarrow e^{2y} = \frac{1+x}{1-x}$$

$$\text{so } e^y = \left(\frac{1+x}{1-x} \right)^{1/2}$$

Since $e^y > 0$, discount
-ve square root

$$\Rightarrow y = \ln \left(\frac{1+x}{1-x} \right)^{1/2}$$

$$= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

So

$$\operatorname{artanh}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

- d) Using (b) and (c), find the exact value of $\int_0^3 \frac{1}{(16-x^2)^2} dx$.

[4 marks]

Let $a=4$, then

$$\begin{aligned}
 \int_0^3 \frac{1}{(16-x^2)^2} dx &= \left[\frac{x}{2x^2(4^2-x^2)} + \frac{1}{2x^2 \times 4} \operatorname{artanh}\left(\frac{x}{4}\right) \right]_0^3 \\
 &= \frac{3}{2 \times 16 \times (16-9)} + \frac{1}{2 \times 16 \times 4} \operatorname{artanh}\left(\frac{3}{4}\right) \\
 &= \frac{3}{224} + \frac{1}{128} \operatorname{artanh}\left(\frac{3}{4}\right) \\
 &= \frac{3}{224} + \frac{1}{128} \frac{1}{2} \ln\left(\frac{7/4}{1/4}\right) \\
 &= \frac{3}{224} + \frac{1}{256} \ln(7)
 \end{aligned}$$

$$\Gamma \approx 0.02099406866$$