

AQA A-Level Further Maths 2023 Paper

1C

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100

Solutions



1 Given $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$ find \mathbf{AB}

$$\begin{pmatrix} 18 & 8 \\ 10 & 4 \end{pmatrix} \quad \begin{pmatrix} 7 & 4 \\ 4 & 3 \end{pmatrix} \quad \begin{pmatrix} 12 & 4 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 14 & 12 \\ 10 & 8 \end{pmatrix}$$

[1 mark]

$$\mathbf{AB} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 \times 4 + 2 \times 3 & 3 \times 2 + 2 \times 1 \\ 1 \times 4 + 2 \times 3 & 1 \times 2 + 2 \times 1 \end{pmatrix} = \begin{pmatrix} 18 & 8 \\ 10 & 4 \end{pmatrix}$$

2 $\frac{3}{1+4i} =$

$$3 + \frac{3}{4}i \quad \frac{3}{17}(1-4i) \quad \frac{3}{17}(1+4i) \quad 1-4i$$

[1 mark]

$$\frac{3}{1+4i} \times \frac{1-4i}{1-4i} = \frac{3(1-4i)}{1+16} = \frac{3}{17}(1-4i)$$

3 What are the asymptotes of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$?

$$y = \pm 3x \quad y = \pm 2x \quad y = \pm \frac{3}{2}x$$

$$y = \pm \frac{2}{3}x$$

[1 mark]

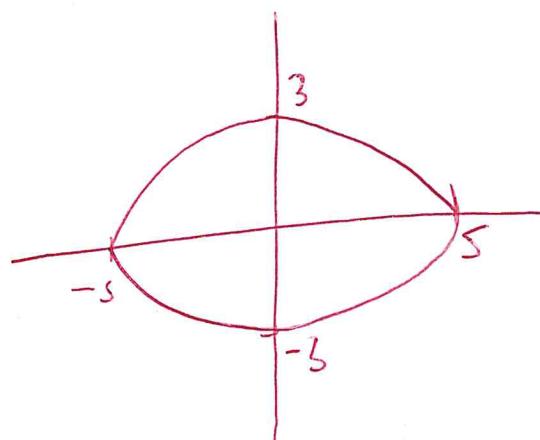
- 4 Sketch the curve $9x^2 + 25y^2 = 225$, indicating any intersections with the axes.

[3 marks]

$$9x^2 + 25y^2 = 225$$

$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$$

Hopgully your ellipse is better than this



- 5 What are the asymptotes of the rational function $y = \frac{3x+1}{6x^2-x-2}$?

[3 marks]

$$y = \frac{3x+1}{6x^2-x-2} = \frac{3x+1}{(2x+1)(3x-2)}$$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = -\frac{1}{2}$ and $x = \frac{2}{3}$

- 6 Find the derivative of $y = \arctan(5x)$, showing full reasoning.

[4 marks]

$$y = \arctan(5x)$$

$$\tan(y) = 5x$$

$$\Rightarrow \frac{\sec^2(y) dy}{dx} = 5$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{\sec^2(y)} \\ = \frac{5}{1 + \tan^2(y)}$$

$$= \frac{5}{1 + 25x^2}$$

- 7 a) Explain why $\frac{3x^4}{2 \ln(1 + x^2) - 2x^2}$ is an indeterminate form when $x = 0$.

[1 mark]

When $x = 0$ we have $\frac{0}{0}$ and so

$\frac{3x^4}{2 \ln(1 + x^2) - 2x^2}$ is an indeterminate form.

- b) Using series expansions find $\lim_{x \rightarrow 0} \frac{3x^4}{2 \ln(1 + x^2) - 2x^2}$

[4 marks]

$$\ln(1 + x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots$$

Hence

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3x^4}{2 \ln(1 + x^2) - 2x^2} &= \lim_{x \rightarrow 0} \frac{3x^4}{2\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots\right) - 2x^2} \\ &= \lim_{x \rightarrow 0} \frac{3x^4}{-\frac{x^4}{1} + \frac{2x^6}{3} - \frac{x^8}{2} + \dots} \\ &= \cancel{\lim_{x \rightarrow 0}} \frac{3}{-1 + \frac{2}{3}x^2 - \frac{x^4}{2} + \dots} \\ &= -3 \end{aligned}$$

- 8 a) Derive a reduction formula for the integral $I_n = \int \tan^n(x) dx$

[4 marks]

Let $\tan^n(x) = \tan^2(x)\tan^{n-2}(x)$ then

$$\begin{aligned} I_n &= \int \tan^n(x) dx \\ &= \int \tan^2(x)\tan^{n-2}(x) dx \\ &= \int (\sec^2(x) - 1)\tan^{n-2}(x) dx \\ &= \int \sec^2(x)\tan^{n-2}(x) dx - \int \tan^{n-2}(x) dx \end{aligned}$$

So

$$I_n = \frac{1}{n-1} \tan^{n-1}(x) - I_{n-2}$$

- b) Using your reduction formula found in (a) evaluate $\int_0^{\frac{\pi}{4}} \tan^4(x) dx$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \tan^4(x) dx &= \left[\frac{1}{3} \tan^3(x) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2(x) dx \\ &= \frac{1}{3} - \left[\tan(x) \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan^2(x) dx \\ &= \frac{1}{3} - 1 + \left[x \right]_0^{\frac{\pi}{4}} \\ &= -\frac{2}{3} + \frac{\pi}{4} \end{aligned}$$

- 9 Prove that $f(n) = 9^n - 2^n$ is divisible by 7 for all $n \in \mathbb{N}$.

[7 marks]

Step 1: When $n=1$,

$$9^1 - 2^1 = 7 \text{ which is divisible by 7.}$$

Step 2: We assume true for $n=k$, i.e. $7 \mid 9^k - 2^k$

Step 3: Consider $f(k+1)$

$$\begin{aligned} f(k+1) &= 9^{k+1} - 2^{k+1} \\ &= 9 \cdot 9^k - 2 \cdot 2^k \end{aligned}$$

$$= \underbrace{9(9^k - 2^k)}_{\substack{\text{divisible by} \\ \text{7 by inductive hypothesis}}} + \underbrace{7 \cdot 2^k}_{\substack{\text{divisible by 7 as 7 is a factor}}}$$

Wence the truth of $f(k)$ implies $f(k+1)$ is true.

Step 4: As we have shown $f(1)$ to be divisible by 7 and if $f(k)$ is divisible by 7, then so is $f(k+1)$, ~~therefore~~ by the principle of mathematical induction $f(n)$ is divisible by all $n \in \mathbb{N}$

- 10 a) Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 2 & -2 \\ -3 & 2 & -4 \end{pmatrix}$, showing all steps.

[4 marks]

Mines:

$$\left(\begin{vmatrix} 2 & -2 \\ 2 & -4 \end{vmatrix} \quad \begin{vmatrix} 4 & -2 \\ -3 & -4 \end{vmatrix} \quad \begin{vmatrix} 4 & 2 \\ -3 & 2 \end{vmatrix} \right) = \begin{pmatrix} -4 & -22 & 14 \\ 14 & -5 & 13 \\ -8 & -8 & -8 \end{pmatrix}$$

$$|A| = 2 \times -4 - 3 \times -22 + 1 \times 14 = 72$$

$$\begin{pmatrix} -4 & 22 & 14 \\ 14 & -5 & 13 \\ -8 & 8 & -8 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 14 & -8 \\ 22 & -5 & 8 \\ 14 & -13 & -8 \end{pmatrix}$$

Hence,

$$A^{-1} = \frac{1}{72} \begin{pmatrix} -4 & 14 & -8 \\ 22 & -5 & 8 \\ 14 & -13 & -8 \end{pmatrix}$$

- b) Prove that $(A^T)^{-1} = (A^{-1})^T$

[2 marks]

$$\begin{aligned}(A^{-1})^T A^T &= (AA^{-1})^T \\ &= I^T \\ &= I\end{aligned}$$

Hence, A^T is the inverse of $(A^{-1})^T$

- c) Hence, write down the inverse of A^T .

[1 mark]

$$\begin{aligned}(A^T)^{-1} &= (A^{-1})^T \\ &= \frac{1}{72} \begin{pmatrix} -4 & 22 & 14 \\ 14 & -5 & -13 \\ -8 & 8 & -8 \end{pmatrix}\end{aligned}$$

11 a) Show that $(2 + 3i)^3 = -46 + 9i$

[2 marks]

$$\begin{aligned}(2 + 3i)^3 &= 2^3 + 3 \times 2^2 \times 3i + 3 \times 2 \times (3i)^2 + (3i)^3 \\&= 8 + 36i - 54 - 27i \\&= -46 + 9i\end{aligned}$$

- b) Using part (a), show that $2 + 3i$ is a root of the polynomial
 $p(z) = z^3 - 8z^2 + 29z - 52$

[3 marks]

$$\begin{aligned}p(2+3i) &= (2+3i)^3 - 8(2+3i)^2 + 29(2+3i) - 52 \\&= -46 + 9i - 8(-5 + 12i) + 29(2+3i) - 52 \\&= -46 + 9i + 40 - 96i + 58 + 87i - 52 \\&= 0\end{aligned}$$

Hence $2 + 3i$ is a root of the polynomial.

- c) Explain why $p(z)$ must cross the x -axis.

[1 mark]

For a polynomial with real ~~coefficient~~, coefficients complex roots occur in conjugate pairs.

Hence, $p(z)$ must have one real root and so must cross the x -axis.

- d) Fully factorise $p(z)$

[2 marks]

$$p(z) = (z^2 - 4z + 13)(z - 4)$$

12 a) The solutions α, β and γ of the cubic equation

$$x^3 - 3x^2 - 13x + 15 = 0$$

form an arithmetic sequence. Solve the equation.

[4 marks]

Let the roots be $\alpha-d, \alpha, \alpha+d$
Then, using Viète's formulae

$$\alpha-d + \alpha + \alpha+d = \frac{-3}{1} = 3$$

$$\text{so } 3\alpha = 3 \quad (1)$$

$$(\alpha-d)\alpha + (\alpha-d)(\alpha+d) + \alpha(\alpha+d) = -13$$

$$\text{so } 3\alpha^2 - d^2 = -13 \quad (2)$$

For $(\alpha-d)(\alpha)(\alpha+d) = -15$ so, $\alpha^3 - ad^2 = -15$

$$\text{From (1)} \quad 3\alpha = 3 \quad \Rightarrow \alpha = 1$$

$$\text{Sub into (2)} \quad 3 \times 1^2 - d^2 = 13$$

$$\Rightarrow \quad 16 = d^2 \\ d = 4$$

Hence, the solutions are

$$x = \alpha - d = -3$$

$$x = \alpha = 1$$

$$x = \alpha + d = 5$$

- b) Find an equation with roots $\alpha - 2, \beta - 2$ and $\gamma - 2$

[3 marks]

$$\text{Let } w = x - 2 \Rightarrow x = w + 2$$

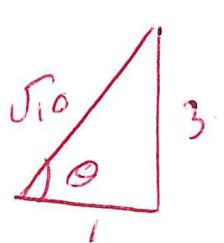
Substitute into the equation

$$(w+2)^3 - 3(w+2)^2 - 13(w+2) + 15 \\ = w^3 + 3w^2 + 3w - 15$$

Hence $x^3 + 3x^2 + 3x - 15$ has roots $\alpha - 2, \beta - 2$ and $\gamma - 2$

- 13 a) Find the invariant points of the transformation representing a reflection in the line $y = 3x$.

[7 marks]



$$\tan(\theta) = 3$$

$$\sin(\theta) = \frac{3}{\sqrt{10}}$$

$$\cos(\theta) = \frac{1}{\sqrt{10}}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$= 2 \times \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{10}}$$

$$= \frac{6}{10}$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

$$= \frac{2}{10} - 1$$

$$= -\frac{8}{10}$$

Now, if M is the matrix representing the transformation

$$M = \begin{pmatrix} -\frac{8}{10} & \frac{6}{10} \\ \frac{6}{10} & -\frac{8}{10} \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

Invariant points when

$$\begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{So } -\frac{4}{5}x + \frac{3}{5}y = x \Rightarrow \frac{3}{5}y = \frac{9}{5}x$$

$$\text{and } \frac{3}{5}x + \frac{4}{5}y = y \Rightarrow -\frac{y}{5} = -\frac{3}{5}x$$

$$\text{Hence } y = 3x$$

So all invariant points are of the form $\begin{pmatrix} x \\ 3x \end{pmatrix}$
and lie on line $y = 3x$ (the mirror line of the transformation).

- b) Without calculating them what do you know about the invariant lines of this transformation?

[2 marks]

One of the form $y = 3x$ and infinitely many lines of the form $y = -\frac{1}{3}x + c$, $c \in \mathbb{R}$

- 14 A particle P moves along a horizontal axes under the action of a force directed towards a fixed point O . The displacement, x metres, of P from its initial position at time t satisfies the differential equation below

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 30$$

Assuming that at time $t = 0$ the particle is moving through the point $x = 3$ with velocity 24 ms^{-1} .

- a) Solve the differential equation to obtain an expression for x in terms of t .

[8 marks]

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 30$$

Auxiliary equation

$$m^2 + 2m + 10 = 0$$

$$\Rightarrow m = -1 \pm 3i$$

Complementary function

$$x_c = e^{-t} (A \cos(3t) + B \sin(3t))$$

For the particular integral try $x_p = C$, then

$$10C = 30$$

$$\Rightarrow C = 3$$

Hence, the general solution is

$$x = e^{-t} (A \cos(3t) + B \sin(3t)) + 3$$

$$x(0) = 3, \text{ so}$$

$$3 = A + 3 \Rightarrow A = 0$$

$$\text{Therefore } x = e^{-t} B \sin(3t) + 3$$

$$\dot{x} = 3B e^{-t} \cos(3t) + B \sin(3t) e^{-t}$$

$$\dot{x}(0) = 2 \text{ and so}$$

$$2 = 3B$$

$$\Rightarrow B = \frac{2}{3}$$

Hence,

$$x(t) = \frac{2}{3} e^{-t} \sin(3t) + 3$$

- b) What kind of damping is present in this system?

[2 marks]

Consider the discriminant of the auxiliary equation

$$m^2 + 2m + 10 = 0$$

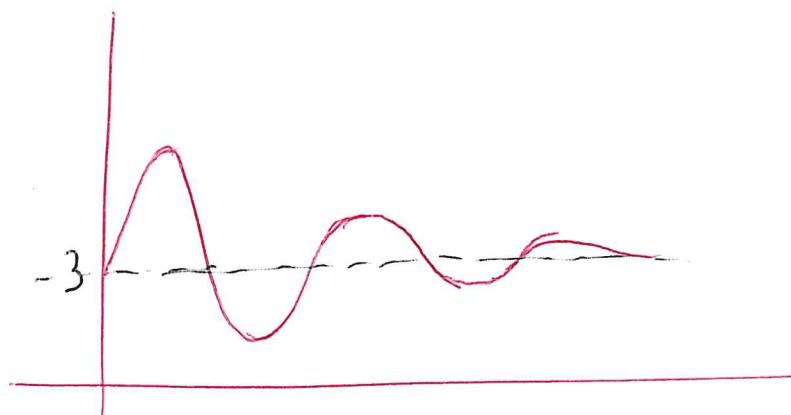
$$\Delta = 2^2 - 4 \times 1 \times 10$$

$$< 0$$

Hence, light damping is present.

- c) Sketch the solution $x(t)$ to the differential equation, clearly showing the initial position and long term behaviour of the motion.

[3 marks]



- 15 a) A curve C has equation $y = 3x^2$. Show that the arc length between $x = 0$ and $x = 1$ can be found by the integral

$$L = \int_0^1 \sqrt{1 + 36x^2} \, dx$$

[2 marks]

$$\text{Arc length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$y = 3x^2 \Rightarrow \frac{dy}{dx} = 6x$$

So

$$L = \int_0^1 \sqrt{1 + (6x)^2} \, dx$$

$$= \int_0^1 \sqrt{1 + 36x^2} \, dx$$

- b) Using a suitable hyperbolic substitution, showing full reasoning,
 show that $L = \frac{1}{12} \left[6\sqrt{37} + \ln \left(6 + \sqrt{37} \right) \right]$ [10 marks]

Consider

$$\int \sqrt{1+36x^2} dx$$

$$\text{Let } x = \frac{1}{6} \sinh(u) \Rightarrow \frac{dx}{du} = \frac{1}{6} \cosh(u)$$

$$\text{so } dx = \frac{1}{6} \cosh(u) du$$

$$\text{When } x=0, u=0$$

$$\begin{aligned} \text{When } x=1, u &= \operatorname{arsinh}(6) \\ &= \ln \left(6 + \sqrt{6^2+1} \right) \\ &= \ln(6+\sqrt{37}) \end{aligned}$$

Then,

$$L = \int_0^{\ln(6+\sqrt{37})} \sqrt{1+36 \times \frac{1}{36} \sinh^2(u)} \cdot \frac{1}{6} \cosh(u) du$$

$$= \frac{1}{6} \int_0^{\ln(6+\sqrt{37})} \sqrt{1+\sinh^2(u)} \cosh(u) du$$

$$= \frac{1}{6} \int_0^{\ln(6+\sqrt{37})} \sqrt{\cosh^2(u)} \cosh(u) du$$

$$\begin{aligned}
 &= \frac{1}{6} \int_0^{\ln(6+\sqrt{37})} \cosh^2(u) du \\
 &= \frac{1}{12} \int_0^{\ln(6+\sqrt{37})} \cosh(2u) + 1 du \quad \text{as } \cosh^2 x = \frac{1}{2} \cosh(2x) + 1 \\
 &= \frac{1}{12} \left[\frac{1}{2} \sinh(2u) + u \right]_0^{\ln(6+\sqrt{37})} \\
 &= \frac{1}{12} \left[\frac{1}{2} \sinh(2\ln(6+\sqrt{37})) + \ln(6+\sqrt{37}) \right]
 \end{aligned}$$

Consider

$$\begin{aligned}
 \sinh(2\ln(6+\sqrt{37})) &= \sinh(\ln((6+\sqrt{37})^2)) \\
 &= \frac{e^{\ln((6+\sqrt{37})^2)} - e^{-\ln((6+\sqrt{37})^2)}}{2} \\
 &= \frac{(6+\sqrt{37})^2 - (6+\sqrt{37})^{-2}}{2} \\
 &= \frac{73+12\sqrt{37} - \frac{1}{73+12\sqrt{37}}}{2} \\
 &= \frac{73+12\sqrt{37} - (73-12\sqrt{37})}{2} = \frac{24\sqrt{37}}{2} = 12\sqrt{37}
 \end{aligned}$$

Hence,

$$L = \frac{1}{12} \left[\frac{12\sqrt{37}}{2} + \ln(6+\sqrt{37}) \right]$$

- 16 a) Find the equation of the plane containing the points $A(1,3,1)$, $B(2,3,3)$ and $C(5,5,4)$.

[4 marks]

$$\vec{AB} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 4 & 2 & 3 \end{vmatrix}$$

$$= \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 13$$

Hence, an equation of the plane is

$$5 \cdot \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix} = 13$$

or in Cartesian form

$$-4x + 5y + 2z = 13$$

- b) Find the shortest distance from the plane found in (a) and the point $D(4,2,1)$.

[7 marks]

Equation of \perp line from b to the plane is

$$\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix}$$

At the intersection point of the line with the plane,

$$\begin{pmatrix} 4 - 4\lambda \\ 2 + 5\lambda \\ 1 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix} = 13$$

Hence,

$$-16 + 16\lambda + 10 + 25\lambda + 2 + 4\lambda = 13$$

$$-4 + 45\lambda = 13$$

$$\text{so } 45\lambda = 17$$

$$\lambda = \frac{17}{45}$$

So, at the intersection point, E , there is the position vector

$$\begin{pmatrix} \frac{112}{45} \\ \frac{45}{45} \\ \frac{35}{45} \\ \frac{79}{45} \end{pmatrix}$$

So shortest distance is

$$d = \left| \frac{68}{4s} \mathbf{i} - \frac{17}{9} \mathbf{j} - \frac{34}{4s} \mathbf{k} \right|$$

$$= \sqrt{\left(\frac{68}{4s}\right)^2 + \left(\frac{17}{9}\right)^2 + \left(\frac{34}{4s}\right)^2}$$

$$= \sqrt{\frac{289}{4s}}$$

$$= \frac{17\sqrt{5}}{4s}$$