

AQA A-Level Further Maths 2023 Paper

1B

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100



1 Evaluate $\sinh(\ln(3))$

$$\frac{4}{3}$$

$$\frac{5}{3}$$

$$\frac{4}{5}$$

$$\ln\left(\frac{4}{3}\right)$$

[1 mark]

$$\begin{aligned}\sinh(\ln(3)) &= \frac{e^{\ln(3)} - e^{-\ln(3)}}{2} \\ &= \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}\end{aligned}$$

2 For the unit vectors \mathbf{i} and \mathbf{j} evaluate $\mathbf{i} \times \mathbf{j} =$

$$\mathbf{i}$$

$$\mathbf{j}$$

$$\mathbf{k}$$

$$-\mathbf{k}$$

[1 mark]

3 Which of the below is the inverse of the matrix $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$?

$$\frac{1}{2} \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} -2 & 1 \\ 4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -3 \end{pmatrix}$$

[1 mark]

$$\det \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = 6 - 4 = 2$$

$$\text{So inverse is } \frac{1}{2} \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

- 4 Find the polar equation of the ellipse with cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

[3 marks]

with $x = r\cos(\theta)$

$y = r\sin(\theta)$

And

$$\frac{(r\cos\theta)^2}{9} + \frac{(r\sin\theta)^2}{4} = 1$$

$$\Rightarrow 4r^2\cos^2\theta + 9r^2\sin^2\theta = 36$$

$$\Rightarrow r^2(4\cos^2\theta + 9\sin^2\theta) = 36$$

$$\Rightarrow r^2 = \frac{36}{4\cos^2\theta + 9\sin^2\theta}$$

- 5 Find, showing all reasoning, $\int_0^3 \frac{9}{\sqrt{x}} dx$

$$\int_0^3 \frac{9}{\sqrt{x}} = \lim_{t \rightarrow 0} \int_t^3 \frac{9}{\sqrt{x}} dx$$

$$= \lim_{t \rightarrow 0} \left[18x^{1/2} \right]_t^3$$

$$= \lim_{t \rightarrow 0} [18\sqrt{3} - 18\sqrt{t}]$$

$$= 18\sqrt{3}$$

- 6 Find the 2×2 matrix representing a rotation, centre the origin of 90° counter clockwise followed by a reflection in the line $y = x$.

[4 marks]

rotation 90° counter clockwise : $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

reflection in the line $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

So the combined transformation has matrix,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- 7 The curve C is obtained by a sequence of transformations of the curve $\frac{x^2}{9} - \frac{y^2}{4} = 1$. First a stretch, scale factor $\frac{1}{2}$ parallel to the x -axis is applied, followed by a translation by the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

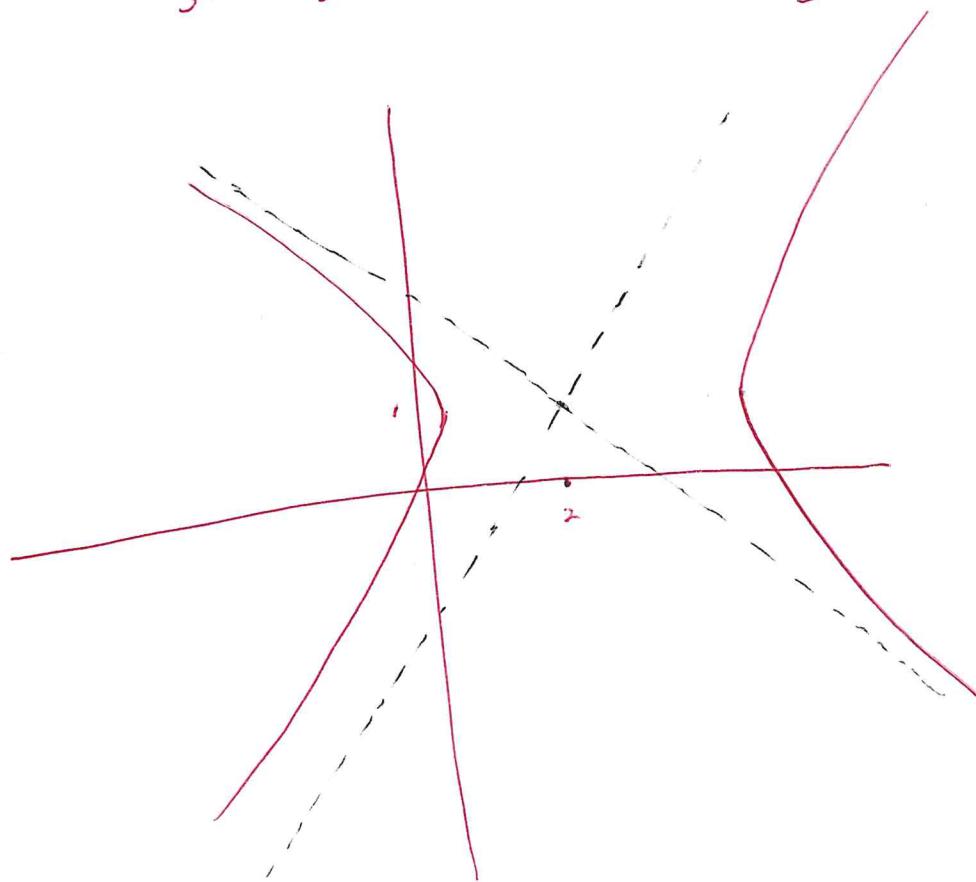
Sketch C , and find the equations of the asymptotes.

[5 marks]

$$\frac{4(x-2)^2}{9} - \frac{(y-1)^2}{4} = 1$$

Asymptotes are $y-1 = \pm \frac{2}{3}(2(x-2))$

$$\text{so } y = \frac{4}{3}x - \frac{5}{3} \quad \text{and} \quad y = -\frac{4}{3}x + \frac{11}{3}$$



8 Find the solution of the differential equation

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3x + 5$$

with initial conditions $y(0) = 2$ and $y'(0) = 4$.

[8 marks]

Auxiliary Equation

$$2m^2 - m - 6 = 0$$

$$(2m+3)(m-2) = 0$$

$$m = -\frac{3}{2} \text{ and } m = 2$$

So C.F. $y = A e^{-\frac{3}{2}x} + B e^{2x}$

For P.I. try $y = Ax + B$
 $y' = A$
 $y'' = 0$

$$-A - 6Ax - 6B = 3x + 5$$

$$\underline{-A} - 6A\underline{x} - 6B = 3 \Rightarrow A = -\frac{1}{2}$$

$$\underline{-A} - 6B = 5 \Rightarrow B = -\frac{3}{4}$$

So P.I. is $y = -\frac{x}{2} - \frac{3}{4}$

and general solution is

$$y = A e^{-\frac{3}{2}x} + B e^{2x} - \frac{x}{2} - \frac{3}{4}$$

$$y' = -\frac{3}{2}A e^{-\frac{3}{2}x} + 2B e^{2x} - \frac{1}{2}$$

$$y(0) = 2, \text{ so}$$

$$A + B - \frac{3}{4} = 2 \Rightarrow A + B = \frac{11}{4} \quad (1)$$

$$y'(0) = 4, \text{ so}$$

$$-\frac{3}{2} A + 2B - \frac{1}{2} = 4$$

$$\Rightarrow -\frac{3}{2} A + 2B = \frac{9}{2} \quad (2)$$

Solving (1) and (2)

$$A = \frac{2}{7}, \quad B = \frac{69}{28}$$

Hence,

$$x(t) = \frac{2}{7} e^{-3t/2} + \frac{69}{28} e^{2t} - \frac{x}{2} - \frac{3}{4}$$

- 9 Sketch the curve $y = \frac{x^2 + 3x + 2}{2x + 3}$ showing the equations of all asymptotes and any intersections of the curve with the coordinate axes.

[7 marks]

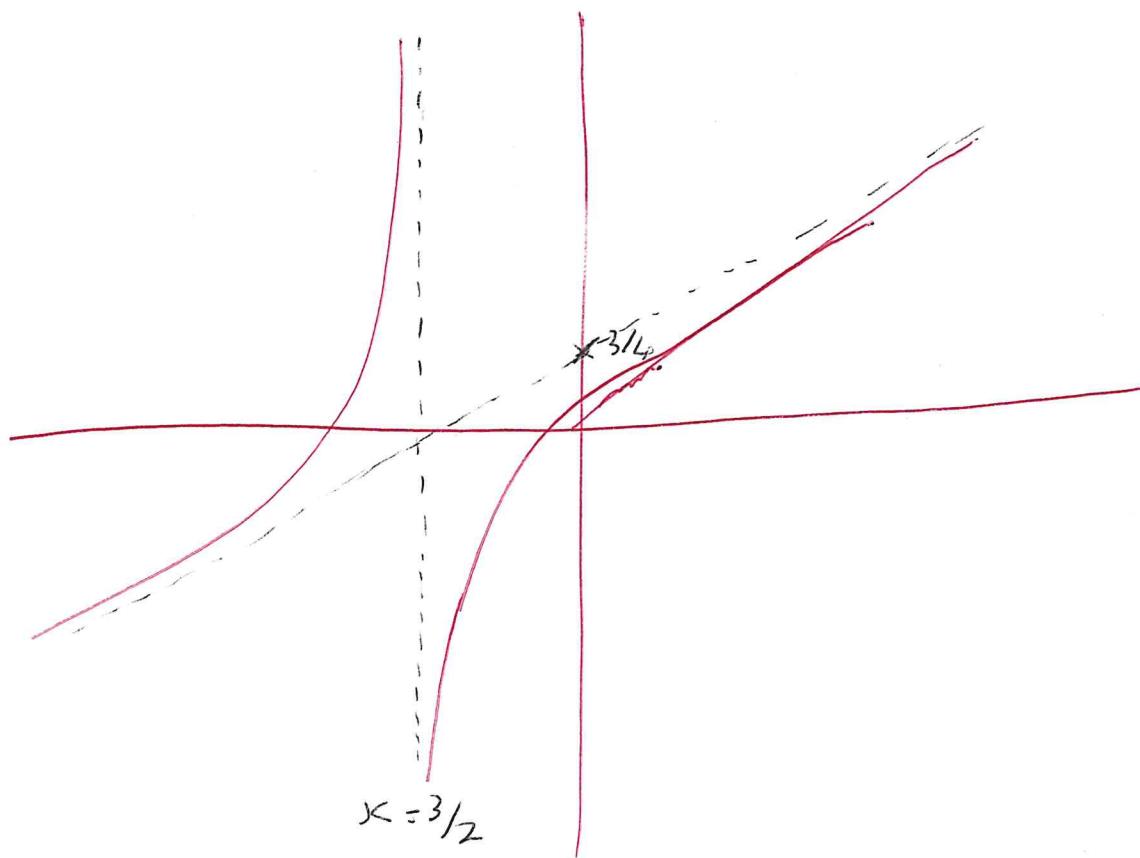
$$\begin{array}{r} \cancel{x^2 + 3x} \\ 2x + 3 \sqrt{x^2 + 3x + 2} \\ \underline{x^2 + \frac{3x}{2}} \\ \frac{3x}{2} + 2 \\ \underline{\frac{3x}{2} + \frac{9}{4}} \\ -\frac{1}{4} \end{array}$$

None, oblique asymptote at $y = \frac{x}{2} + \frac{3}{4}$

$$y = \frac{x}{2} + \frac{3}{4}$$

Vertical asymptote $x = -\frac{3}{2}$

Intersections $(-2, 0), (-1, 0), (0, 2/3)$

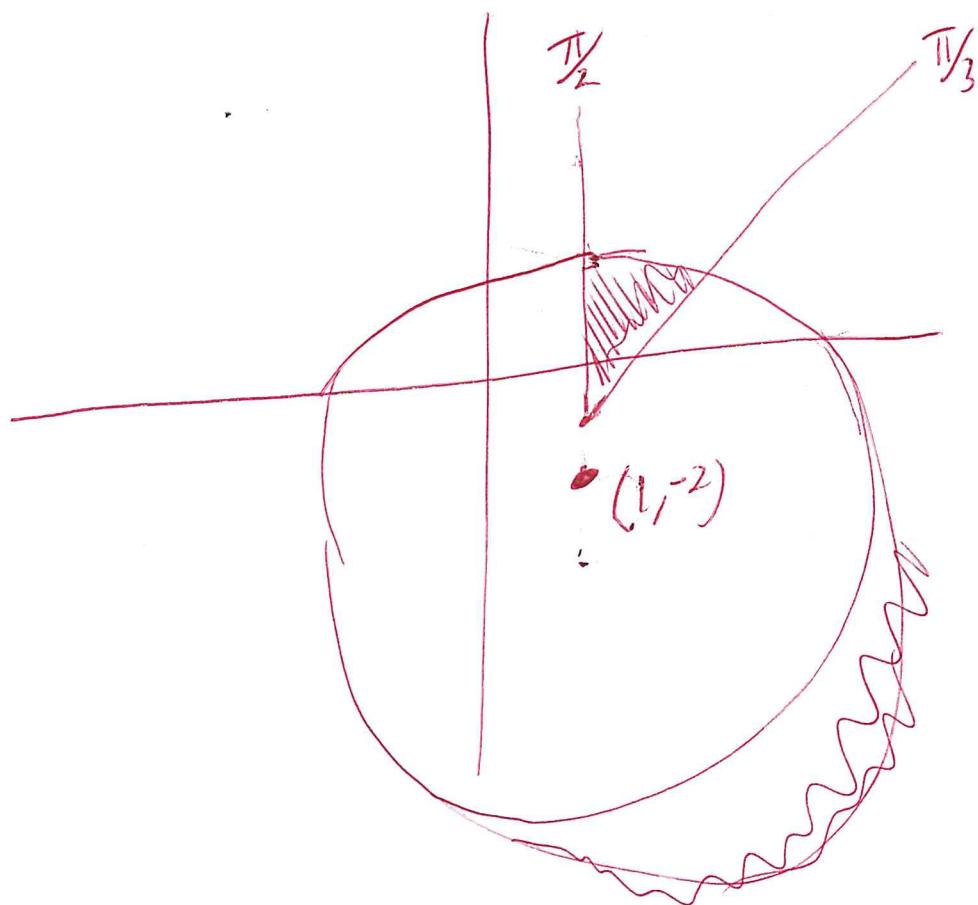




- 10 On an Argand diagram shade the region given by

$$\{z : |z - 1 + 2i| < 4\} \cap \left\{ \frac{\pi}{3} \leq \arg(z - (1 + i)) \leq \frac{\pi}{2} \right\}$$

$|z - 1 + 2i| < 4 \Rightarrow |z - (1 + 2i)| < 4$ [5 marks]
 $\arg(z - (1 + 2i)) \Rightarrow$ half line, starting from (1)



- 11 By considering a suitable volume of revolution prove that the volume of a sphere, of radius r , is $\frac{4}{3}\pi r^3$.

[6 marks]

Consider the function $y = \sqrt{r^2 - x^2}$

Then

$$V = \pi \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx$$

$$= \pi \int_{-r}^r r^2 - x^2 dx$$

$$= \pi \left[r^2x - \frac{x^3}{3} \right]_{-r}^r$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right]$$

$$= \pi \left[2r^3 - \frac{2r^3}{3} \right]$$

$$= \frac{4}{3}\pi r^3$$

- 12 a) Find the general solution to the first order differential equation

$$x \frac{dy}{dx} - 2y = 3x^2$$

[4 marks]

($\div x$)

$$\frac{dy}{dx} - \frac{2}{x} y = 3x$$

$$\text{Integrating factor: } e^{\int -\frac{2}{x} dx} = e^{-2\ln(x)} = \frac{1}{x^2}$$

Then,

$$\frac{d}{dx} \left(\frac{y}{x^2} \right) = \frac{3}{x}$$

$$\Rightarrow \int \frac{d}{dx} \left(\frac{y}{x^2} \right) dx = \int \frac{3}{x} dx$$

$$\Rightarrow \frac{y}{x^2} = 3\ln(x) + C$$

$$\Rightarrow y = 3x^2 \ln(x) + Cx^2$$

- b) Given that $y(1) = 2$, find any constants of integration in your solution to part (a).

$$2 = 3 \cdot 1^2 \ln(1) + C \cdot 1^2$$

$$\Rightarrow C = 2$$

$$\text{So solution is } y = 3x^2 \ln(x) + 2x^2$$

- c) Use Euler's method with a step size of 0.1 to approximate the value of solution $y(1.1)$ to the differential equation $x \frac{dy}{dx} - 2y = 3x^2$ where $y(1) = 2$.

[3 marks]

$$y_{n+1} = y_n + h f(x_n) \text{ for } \frac{dy}{dx} = f(x)$$

$$x \frac{dy}{dx} - 2y = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3x^2 + 2y}{x}$$

$\underbrace{3x^2 + 2y}_f(x)$

$$\begin{aligned} y(1.1) &= y(1) + 0.1 \times f(1) \\ &= 2 + 0.1 \times 5 \\ &= 2.5 \end{aligned}$$

- d) With reference to your solution found in parts (a) and (b) comment on the accuracy of Euler's method in this case.

Exact solution gives $y(1.1) = 3 \times 1.1^2 \times \ln(1.1) + 2 \times 1.1^2$ [1 mark]

$$\begin{aligned} &= 2.765975953 \end{aligned}$$

Not the strongest of approximations

13 Prove, by induction, that

$(\cosh(\theta) + \sinh(\theta))^n \equiv \cosh(n\theta) + \sinh(n\theta)$

Step 1: Let $P(n)$ be the statement $= (\cosh \theta + \sinh \theta)^n = \cosh(n\theta) + \sinh(n\theta)$

When $n=1$

$$\text{LHS: } (\cosh \theta + \sinh \theta)^1 = \cosh \theta + \sinh \theta$$

$$\text{RHS: } \cosh(1 \times \theta) + \sinh(1 \times \theta) = \cosh(\theta) + \sinh(\theta)$$

so $P(1)$ is true.

Step 2: We assume $P(k)$ is true, i.e

$$(\cosh(\theta) + \sinh(\theta))^k = \cosh(k\theta) + \sinh(k\theta)$$

Step 3:

$$\begin{aligned} (\cosh(\theta) + \sinh(\theta))^{k+1} &= (\cosh(\theta) + \sinh(\theta))^k (\cosh(\theta) + \sinh(\theta)) \\ &= (\cosh(k\theta) + \sinh(k\theta)) (\cosh(\theta) + \sinh(\theta)) \\ &= \cosh(k\theta) \cosh(\theta) + \cosh(k\theta) \sinh(\theta) \\ &\quad + \sinh(k\theta) \cosh(\theta) + \sinh(k\theta) \sinh(\theta) \\ &= \cosh((k+1)\theta) + \sinh((k+1)\theta) \end{aligned}$$

by hyperbolic addition
formulae.

Step 4: We have shown the statement $P(1)$ is true
and if $P(k)$ is true then so is $P(k+1)$. Hence, by
the principle of mathematical induction it is
true for all n .

14 Find, in factorised form $\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & 1 \\ a^3 & b^3 & 1 \end{vmatrix}$.

[6 marks]

$$\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & 1 \\ a^3 & b^3 & 1 \end{vmatrix} = a \begin{vmatrix} 1 & b & 1 \\ a^2 & b^2 & 1 \\ a^2 & b^3 & 1 \end{vmatrix}$$

$$= ab \begin{vmatrix} 1 & 1 & 1 \\ a & b & 1 \\ a^2 & b^2 & 1 \end{vmatrix}$$

$$= ab \begin{vmatrix} 0 & 1 & 1 \\ a^{-1} & b & 1 \\ a^{2-1} & b^2 & 1 \end{vmatrix}$$

$$= ab(a-1) \begin{vmatrix} 0 & 1 & 1 \\ 1 & b & 1 \\ a+1 & b^2 & 1 \end{vmatrix}$$

$$= ab(a-1) \begin{vmatrix} 0 & 0 & 1 \\ 1 & b-1 & 1 \\ a+1 & b^2-1 & 1 \end{vmatrix}$$

$$= ab(a-1)(b-1) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ a+1 & b+1 & 1 \end{vmatrix}$$

$$= ab(a-1)(b-1)(b-a)$$

- 15 a) Find, in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$ the vector equation of the line, l_1 , which passes through the points $A(2, 1, 5)$ and $B(3, 3, 3)$.
[3 marks]

$$\vec{OA} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \quad \vec{OB} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\vec{AB} = -\vec{OA} + \vec{OB} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

Hence,

$$\mathcal{L} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

- b) Find the shortest distance between the line l_1 and the line l_2 with equation $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

[6 marks]

$$\mathcal{L}_1 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \mathcal{L}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

Let P be on \mathcal{L}_1 , then $\vec{OP} = \begin{pmatrix} 2+\lambda \\ 1+2\lambda \\ 5-2\lambda \end{pmatrix}$

Let Q be on \mathcal{L}_2 then $\vec{OQ} = \begin{pmatrix} -2+3\mu \\ 1+\mu \\ 3-2\mu \end{pmatrix}$

$$\text{Then } \vec{PQ} = \begin{pmatrix} -4+3\mu-\lambda \\ \mu-2\lambda \\ -2-2\mu+2\lambda \end{pmatrix}$$

$$\vec{PQ} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0 \Rightarrow -4 + 3\mu - \lambda + 2\mu - 4\lambda + 4\tau 4\mu - 4\lambda = 0$$

$$\Rightarrow 9\mu - 9\lambda = 0 \quad \textcircled{1}$$

$$\vec{PQ} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \Rightarrow -12 + 9\mu - 3\lambda + \mu - 2\lambda + 4 + 4\mu - 4\lambda = 0$$

$$\Rightarrow -8 + 14\mu - 9\lambda = 0$$

$$14\mu - 9\lambda = 8 \quad \textcircled{2}$$

Solving \textcircled{1} and \textcircled{2} $\mu = \frac{8}{5}, \lambda = \frac{8}{5}$

Now, $\vec{PQ} = \begin{pmatrix} 4/5 \\ -8/5 \\ -2 \end{pmatrix}$

$$\text{So } |\vec{PQ}| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{-8}{5}\right)^2 + (-2)^2}$$

$$= \frac{6\sqrt{5}}{5}$$

16 a) Find $\int \frac{5x^2 + 5x + 48}{(x+2)(x^2+25)} dx$

[7 marks]

Partial fractions of the form $\frac{A}{x+2} + \frac{Bx+C}{x^2+25}$

S_0

$$\int \frac{5x^2 + 5x + 48}{(x+2)(x^2+25)} dx = \int \frac{2}{x+2} + \frac{3x}{x^2+25} - \frac{1}{x^2+25}$$

$$= 2 \ln|x+2| + \frac{3}{2} \ln|x^2+25| - \frac{1}{5} \arctan\left(\frac{x}{5}\right)$$



- b) Using the midpoint rule with two strips of width 1, find the length of the curve $y = \frac{5x^2 + 5x + 48}{(x+2)(x^2 + 25)}$ between $x = 2$ and $x = 4$.

[7 marks]

$$\frac{dy}{dx} = \frac{-2x(3x-1)}{(x^2+25)^2} + \frac{3}{x^2+25} - \frac{2}{(x+2)^2}$$

$$S = \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

ordinates

x	y
2.5	1.000649426
3.5	1.00056112

Then

$$S = \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= h(y_{2.5} + y_{3.5})$$

$$= \frac{4-2}{2} (1.000649426 + 1.00056112)$$

$$= 2.001210566$$

≈ 2.001 units



17 Let α, β and γ be the roots of the cubic polynomial $p(x) = x^3 - x^2 - 17x - 15$.

a) Find $\alpha^2 + \beta^2 + \gamma^2$.

[2 marks]

$$\alpha + \beta + \gamma = 1 \quad \alpha\beta + \alpha\gamma + \beta\gamma = -17 \quad \alpha\beta\gamma = 15$$

$$(\alpha + \beta + \gamma)^2 = (\alpha^2 + \beta^2 + \gamma^2) + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\Rightarrow 1^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(-17)$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 35$$

b) Find $\alpha^3 + \beta^3 + \gamma^3$.

[3 marks]

$$\alpha^3 = \alpha^2 + 17\alpha + 15$$

$$\beta^3 = \beta^2 + 17\beta + 15$$

$$\gamma^3 = \gamma^2 + 17\gamma + 15$$

$$\begin{aligned} \alpha^3 + \beta^3 + \gamma^3 &= (\alpha^2 + \beta^2 + \gamma^2) + 17(\alpha + \beta + \gamma) + 45 \\ &= 35 + 17 \times 1 + 45 \\ &= 97 \end{aligned}$$

- c) Find the polynomial with roots $\alpha - 1$, $\beta - 1$ and $\gamma - 1$.

Let $w = \text{transf } x - 1$

then

$$x = w+1$$

$$(w+1)^3 - (w+1)^2 - 17(w+1) - 15 = w^3 + 2w^2 - 16w - 32$$

So the polynomial $x^3 + 2x^2 - 16x - 32$ has roots
 $\alpha - 1, \beta - 1, \gamma - 1$