1) $f(x) = 9x^3 - 33x^2 - 55x - 25$

$= (x-5)(9x^2 + 12x + 5)$

$9x^2 + 12x + 5 = 0$

$a = 9$
$b = 12$
$c = 5$

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 5 \times 9}}{18}$$

$$= \frac{-12 \pm \sqrt{-36}}{18}$$

$$= \frac{-12 \pm 6i}{18}$$

$$= -\frac{2}{3} \pm \frac{i}{3}$$

So $x = 5$ or $\frac{1}{3}(-2-i)$ or $\frac{1}{3}(-2+i)$

2) $3 + x\sin\left(\frac{x}{4}\right) = 0$ where x is measured in radians

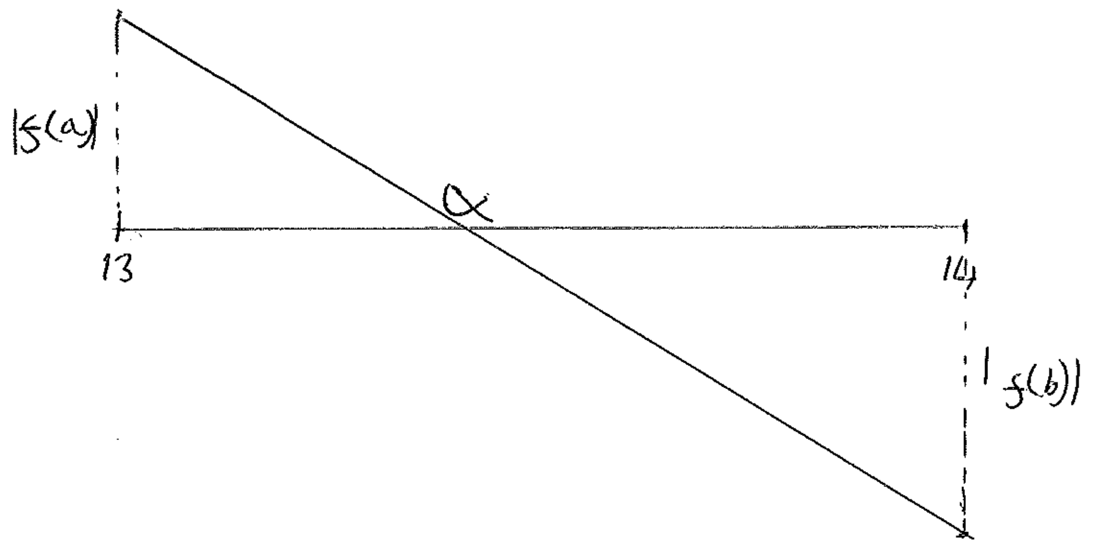
a)

a	b	f(a)	f(b)	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
13	14	1.59346325	-1.910965188	13.5	-0.1224664674
13	13.5	1.59346325	-0.1224664674	13.25	0.7464857641

so interval of width 0.25 is [13.25, 13.5]

b) $a = 13$, $f(a) = 1.59346325$

$b = 14$, $f(b) = -1.910965188$



so $\dfrac{\alpha - 13}{14 - \alpha} = \dfrac{|f(a)|}{|f(b)|}$

so $|f(b)|(\alpha - 13) = |f(a)|(14 - \alpha)$

$\Rightarrow \alpha = \dfrac{14|f(a)| + 13|f(b)|}{|f(b)| + |f(a)|} = 13.4546999$

$= 13.455$

3) a)

$$\sum_{r=1}^{n}(r+1)(r+4) = \sum_{r=1}^{n} r^2+5r+4$$

$$= \sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 1$$

$$= \frac{n}{6}(n+1)(2n+1) + \frac{5n}{2}(n+1) + 4n$$

$$= \frac{n}{6}(n+1)(2n+1) + \frac{15n}{6}(n+1) + 24n$$

$$= \frac{n}{6}\left[(n+1)(2n+1) + 15n + 15 + 24\right]$$

$$= \frac{n}{6}\left[2n^2+3n+1+15n+15+24\right]$$

$$= \frac{n}{6}\left[2n^2+18n+40\right.$$

$$= \frac{n}{3}\left[n^2+9n+20\right]$$

$$= \frac{n}{3}(n+4)(n+5)$$

b)

$$\sum_{r=n+1}^{2n}(r+1)(r+4) = \sum_{r=1}^{2n}(r+1)(r+4) - \sum_{r=1}^{n}(r+1)(r+4)$$

$$= \frac{2n}{3}(2n+4)(2n+5) - \frac{n}{3}(n+4)(n+5)$$

$$= \frac{n}{3}\left[2(2n+4)(2n+5) - (n+4)(n+5)\right]$$

$$= \frac{n}{3}\left[8n^2+36n+40-n^2-9n-20\right]$$

$$= \frac{n}{3}\left[7n^2+27n+20\right]$$

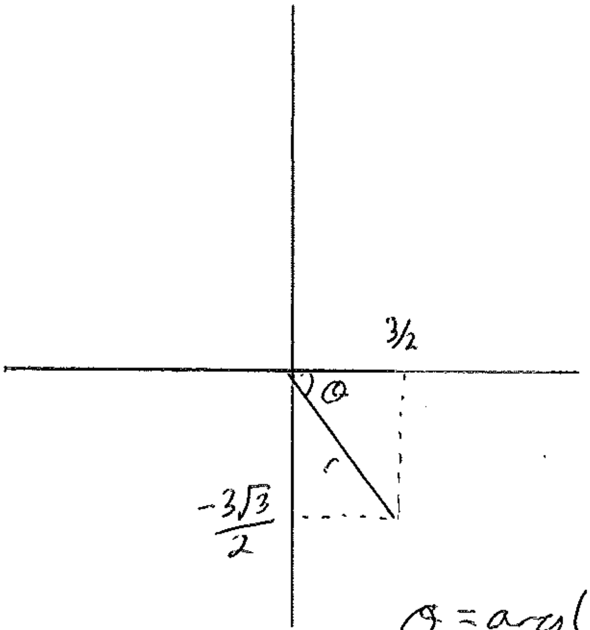
$$= \frac{n}{3}(n+1)(7n+20) \qquad \Rightarrow a=7,\ b=20$$

4) a) $z_2 = \dfrac{6}{1+i\sqrt{3}} \times \dfrac{1-i\sqrt{3}}{1-i\sqrt{3}}$

$= \dfrac{6 - 6\sqrt{3}i}{1+3}$

$= \dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i$

b)



$r = |z_2|$

$= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{9+27}{4}} = \sqrt{\frac{36}{4}}$

$= \sqrt{9}$

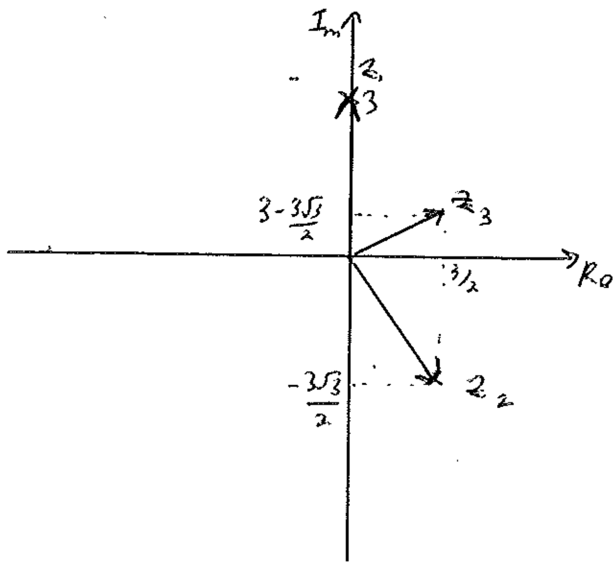
$= 3$

$\theta = \arg(z_2) = -\tan^{-1}\left(\dfrac{+\frac{3\sqrt{3}}{2}}{3/2}\right)$

$= -\tan^{-1}(\sqrt{3})$

$= -\dfrac{\pi}{3}$

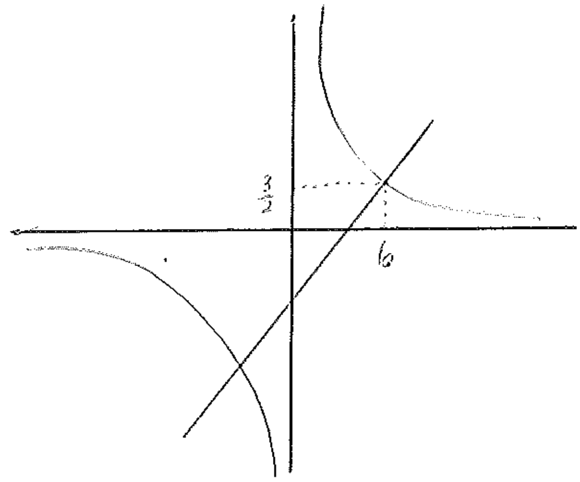
c) $z_1 + z_2 = \dfrac{3}{2} + \left(3 - \dfrac{3\sqrt{3}}{2}\right)i$



5) $xy = 9$ A is Point $\left(6, \frac{3}{2}\right)$ on H.

from point A $\quad ct = 6$ and $\frac{c}{t} = \frac{3}{2} \Rightarrow c = 3,\ t = 2$ at A.

a) $y = \frac{9}{x} = 9x^{-1}$



$\frac{dy}{dx} = -\frac{9}{x^2}$

When $x = 6$ $\quad \frac{dy}{dx} = \frac{-9}{36} = -\frac{1}{4}$

so gradient of normal is 4

Normal has equation $\quad y = 4x + c$ and passes through the point $(6, 3/2)$ so

$$\frac{3}{2} = 4 \times 6 + c \quad \Rightarrow c = \left(24 - \frac{3}{2}\right)$$

$$= -\frac{45}{2}$$

$$\therefore \quad y = 4x - \frac{45}{2}$$

$$\Rightarrow 2y - 8x + 45 = 0$$

b) From the equation of normal $\quad y = 4x - \frac{45}{2}$

$$x\left(4x - \frac{45}{2}\right) = 9$$

$$4x^2 - \frac{45x}{2} = 9$$

$$8x^2 - 45x - 18 = 0$$

$$\Rightarrow x = -\frac{3}{8} \text{ or } 6$$

At $x = \frac{-3}{8}$, $y = -24$ $\quad$ so normal meets the curve again at $\left(\frac{-3}{8}, -24\right)$

b) i) When $n=1$

$$LHS = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5-1) & 5^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$$

Assume ~~true~~ for $n=k$, then for $n=k+1$

$$\begin{pmatrix} +1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) - 5^k & 5.5^k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k+4.5^k-1) & 5.5^k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1}-1) & 5^{k+1} \end{pmatrix}$$

So true for $n=1$, and if assumed true for $n=k$, statement for $n=k+1$ follows. Hence proved for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

ii) When $n=1$

$$LHS = (2.1-1)^2 = 1$$

$$RHS = \frac{1}{3}.1.(4-1) = \frac{3}{3} = 1$$

Assume true for $n=k$, ie $\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(4k^2-1)$

Then for $n=k+1$

$$\sum_{r=1}^{k+1}(2r-1)^2 = \sum_{r=1}^{k}(2r-1)^2 + (2(k+1)-1)^2$$

$$= \frac{1}{3}k(4k^2-1) + (2k+1)^2$$

$$= \frac{1}{3}\left[4k^3 - k + 3(4k^2 - 4k + 1)\right]$$

$$= \frac{1}{3}\left[4k^3 + 12k^2 + 13k + 3\right]$$

$$= \frac{1}{3}(k+1)\left[4k^2 + 8k + 3\right]$$

$$= \frac{1}{3}(k+1)(4(k^2+2k+1)-1)$$

$$= \frac{1}{3}(k+1)(4(k+1)^2-1)$$

So true for $n=1$ and if assumed true for $n=k$, statement for $n=k+1$ follows, hence by the principle of mathematical induction proved true for all $n \in \mathbb{Z}^+$

7i) As singular $\det A = 0$

$$\Rightarrow 5k(k+1) - -3(3k-1) = 0$$

$$5k^2 + 5k + 9k - 3 = 0$$

$$5k^2 + 14k - 3 = 0$$

$$(5k-1)(k+3) = 0$$

so $k = \frac{1}{5}$ or -3.

ii) a) $$B^{-1} = \frac{1}{\det B}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$$

$$= \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$$

b) Vertices of T are

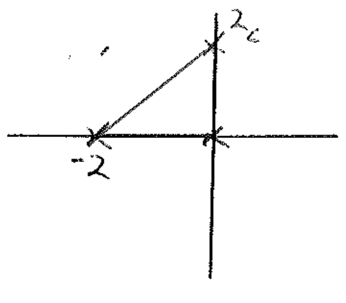
$$= \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}\begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix}$$

T has vertices $(0,0)$, $(-2,0)$, $(0, 2c)$

c) Area of image = $\det(B) \times$ area of object

$$\Rightarrow \text{area of object} = \frac{135}{45} = 3$$



$$3 = \frac{1}{2} \times 2 \times 2c$$

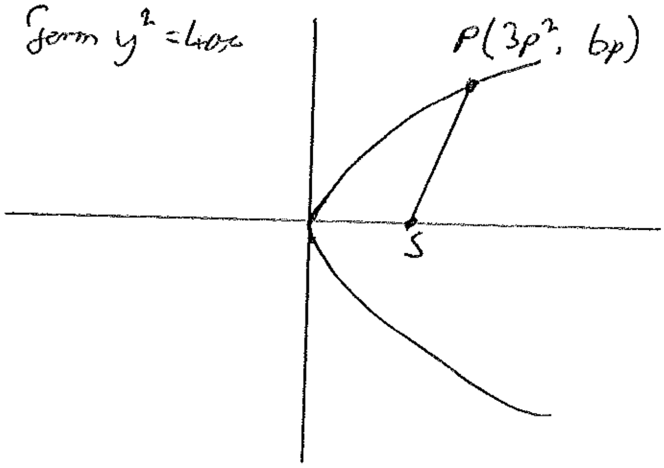
$$6 = +4c$$

$$\Rightarrow c = \frac{3}{2}$$

8) a) Parabola $y^2 = 12x$, form $y^2 = 4ax$

$\Rightarrow a = 3$

so focus S at $(3, 0)$



$\therefore \quad SP = \sqrt{(3p^2-3)^2 + (6p)^2}$

$= \sqrt{9p^4 - 18p^2 + 9 + 36p^2}$

$= \sqrt{9(p^4 + 2p^2 + 1)}$

$= 3\sqrt{(p^2+1)^2}$

$= 3(1+p^2)$

b) $y^2 = 12x \quad \Rightarrow y = 2\sqrt{3}x^{1/2}$

so $\dfrac{dy}{dx} = \sqrt{3}x^{-1/2} = \dfrac{\sqrt{3}}{x^{1/2}}$

For P, gradient of tangent is $\dfrac{\sqrt{3}}{\sqrt{3}p} = \dfrac{1}{p}$

At Q gradient of tangent is $\dfrac{\sqrt{3}}{\sqrt{3}q} = \dfrac{1}{q}$

So equation of tangent at P satisfies

$y = \dfrac{1}{p}x + c$ with $6p = \dfrac{1}{p} \cdot 3p^2 + c \Rightarrow c = 3p$

ie $y = \dfrac{1}{p}x + 3p$

And equation of tangent at Q satisfies

$y = \dfrac{1}{q}x + c$ with $6q = \dfrac{1}{q} \cdot 3q^2 + c \Rightarrow c = 3q$ so $y = \dfrac{1}{q}x + 3q$

They meet at

$$\frac{1}{p}x + 3p = \frac{1}{q}x + 3q$$

$$\Rightarrow x\left(\frac{1}{p} - \frac{1}{q}\right) = 3q - 3p$$

$$x\left(\frac{q-p}{pq}\right) = 3(q-p)$$

$$\Rightarrow x = \frac{3(q-p)}{\frac{q-p}{pq}}$$

$$= 3pq$$

When $x = 3pq$ $\quad y = \frac{1}{p}x + 3p$

$$= 3q + 3p$$

So meet at point $(3pq, 3q+3p)$

$$SR^2 = (3pq-3)^2 + (3q+3p)^2$$

$$= 9p^2q^2 - \cancel{18pq} + 9 + 9q^2 + \cancel{18qp} + 9p^2$$

$$= 9(p^2q^2 + q^2 + p^2 + 1)$$

$$= 9(1+p^2)(1+q^2)$$

$$= 3(1+p^2)3(1+q^2)$$

$$= SP.SQ$$