AQA Paper 2 2019 "Predicted Content"

SOLUTIONS

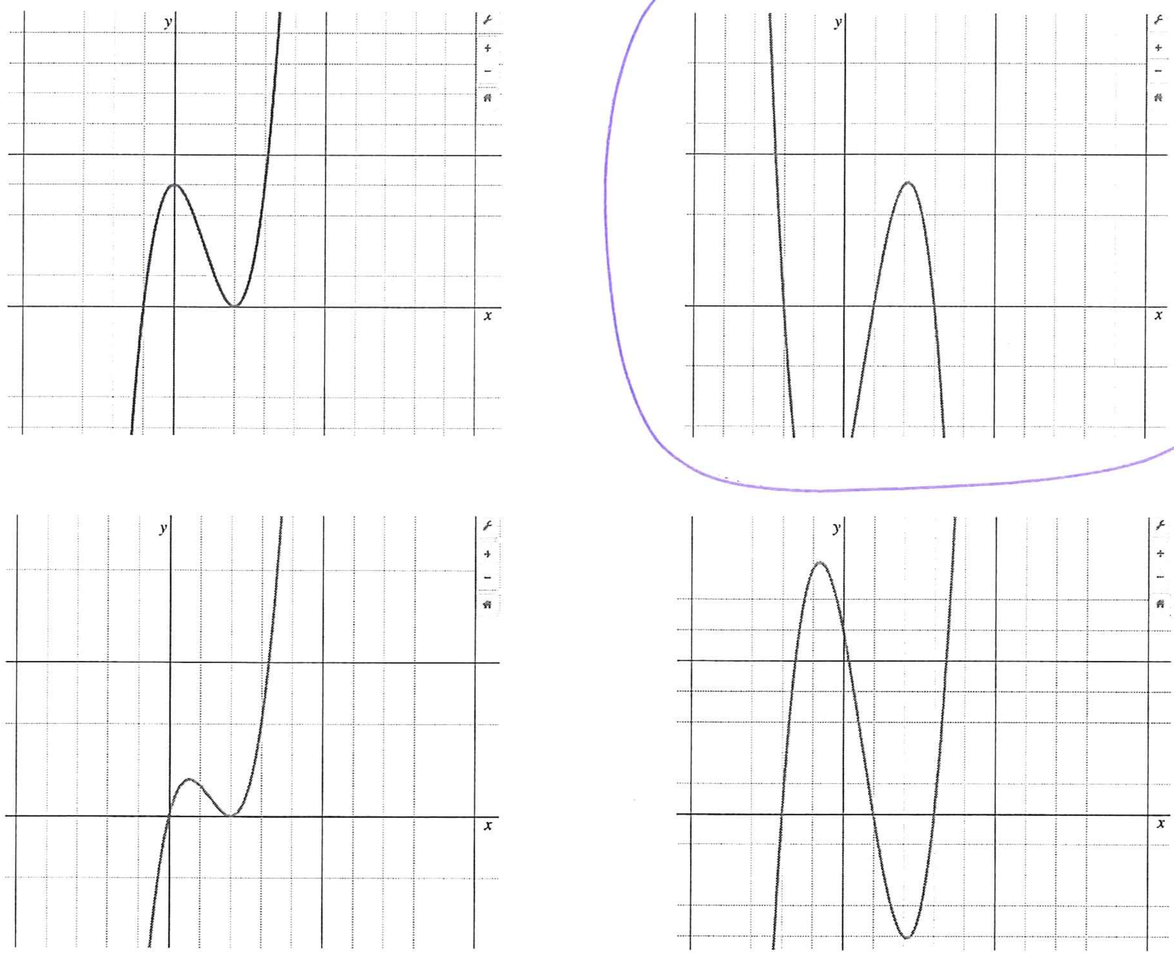
Name: ....

Class: ....

Mark: / 112

1 The function $f(x)$ is cubic with two positive roots and the coefficient of x^3 is ~~positive~~ negative. Which of the graphs below could be $f(x)$?

[1 mark]



2 The mass m of a radioactive substance decays following the equation $m = m_0 e^{-At}$. Justin wants to use a straight line to determine A. To do this he should plot,

[1 mark]

a) m against t

b) $\ln(m)$ against $\ln(t)$

c) $\ln(m)$ against t

d) m against $\ln(t)$

3 Using the quotient rule show that $\frac{d}{dx}\left(\cot(x)\right) = -\operatorname{cosec}^2(x)$

[4 marks]

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Using the Quotient rule,

$$\frac{d}{dx}\cot(x) = \frac{\sin(x)\frac{d}{dx}\left(\cos(x)\right) - \cos(x)\frac{d}{dx}\left(\sin(x)\right)}{\sin^2 x}$$

$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$= \frac{-\left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)}$$

$$= -\frac{1}{\sin^2(x)}$$

$$= -\operatorname{cosec}^2(x)$$

4 Tom claims that for $n \geq 2$, $2^n - 1$ is prime,
Decide if this claim is true. If true provide a proof, or if you decide that it is not true explain how you know.

[3 marks]

It is not true since

$2^2 - 1 = 3$ [prime]

$2^3 - 1 = 7$ [prime]

$2^4 - 1 = 15$ [not prime]

So

5 **a)** Given that, for $f(x) = x^3 + 5x^2 + 8x + 4$, $f(-2) = 0$, factorise $f(x)$ fully.

$f(-2) = 0 \Rightarrow (x+2)$ is a factor,

[2 marks]

and so

$f(x) = (x+2)(x^2+3x+2)$

$= (x+2)^2(x+1)$

b) Hence find $\displaystyle\int \frac{5x^2+16x+13}{x^3+5x^2+8x+4}\,\mathrm{d}x$

[6 marks]

Partial fractions

$$\frac{5x^2+16x+13}{x^3+5x^2+8x+4} = \frac{2}{x+1} + \frac{3}{x+2} - \frac{1}{(x+2)^2}$$

Hence,

$$\int \frac{5x^2+16x+13}{x^3+5x^2+8x+4}\,dx = \int \frac{2}{x+1} + \frac{3}{x+2} - \frac{1}{(x+2)^2}$$

$$= 2\ln|x+1| + 3\ln|x+2| + \frac{1}{x+2}$$

6 A graphic designer is modelling the profile of the roof of a circus tent with the function $y = e^{\frac{x}{2}} + 1$.

a) Describe a sequence of transformations to obtain the graph of $f(x) = e^{\frac{x}{2}} + 1$ from that of $g(x) = e^{x}$.

Parallel to x axis. **[2 marks]**

Stretch scale factor 2 followed by a translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b) Sketch $f(x)$, showing any intersection points with the axes.

[2 marks]

c) Explain, fully justifying your answer, why $f(x)$ has no stationary point.

[3 marks]

$f'(x) = \dfrac{e^{x/2}}{2}$

which is always positive, so no solutions to $\dfrac{dy}{dx} = 0$

d) The designer wishes to make a 3D model of this roof using foam. To do this he rotates the curve $y = e^{\frac{x}{2}} + 1$ about the $x-$axis. The volume of foam required is given by $\pi \int_0^4 (f(x))^2 \, dx$.

Use the trapezium rule with 4 strips to approximate the volume of foam required.

[5 marks]

	x vals	y vals	
x_0	0	4	$\leftarrow y_0$
x_1	1	7.01572637	$\leftarrow y_1$
x_2	2	13.82561976	$\leftarrow y_2$
x_3	3	30.04891506	$\leftarrow y_3$
x_4	4	70.37626223	$\leftarrow y_4$

So $I = \pi \int_0^4 (e^{x/2} + 1)^2 \, dx$

$\simeq \pi \times \frac{1}{2} \times 1 \times \{y_0 + y_4 + 2(y_1 + y_2 + y_3)\}$

$= 276.7064239$

≈ 276.7

e) Will the Trapezium rule produce an under estimate or over estimate of the true volume in this case. Explain your answer.

[2 marks]

Always an over estimate since $f(x)$ is convex.

7 A circle passes through the three points $A(6, -1)$, $B(4, -5)$ and $C(2,1)$.

a) Find the equation of the circle, clearly stating the centre and radius of the circle.

[7 marks]

$\perp$ bisector of AB

$$y = -\frac{1}{2}x - \frac{1}{2} \quad (1)$$

$\perp$ bisector of AC

$$y = 2x - 8 \quad (2)$$

$\perp$ bisector of BC

$$y = \frac{1}{3}x - 3$$

Solve (1) and (2) simultaneously to get

$$\begin{cases} x = 3 \\ y = -2. \end{cases}$$

Centre of circle is $(3, -2)$, call this D

$$|AD| = \sqrt{10}$$

Hence equation of circle is

$$(x-3)^2 + (y+2)^2 = 10$$

b) The tangent to the circle at the point $D(4,1)$ crosses the $x-$axis at E and the $y-$axis at F. Find the area of the triangle OEF.

[5 marks]

Equation of tangent is

$$y = -\frac{1}{3}x + \frac{7}{3}$$

Intersects x axis at 7, and intersects y axis at $\frac{7}{3}$.

Hence area of triangle is $\frac{1}{2} \times \frac{7}{3} \times 7$

$= \frac{49}{6}$ square units

8 Show that $\int_1^4 x\sqrt{3x+1}\ \mathrm{d}x = \frac{4}{135}\left(221\sqrt{13} - 18\right)$

[5 marks]

$I = \int_1^4 x\sqrt{3x+1}\,dx$

Let $u = 3x+1$

$\frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$

$x = \frac{u-1}{3}$

when $x = 1, u = 4$

$x = 4, u = 13$

Hence

$I = \int_4^{13} \left(\frac{u-1}{3}\right)\sqrt{u}\ \frac{du}{3}$

$= \frac{1}{9}\int_4^{13} (u-1)\sqrt{u}\,du$

$= \frac{1}{9}\int_4^{13} u^{3/2} - u^{1/2}\,du$

$= \frac{1}{9}\left[\frac{2}{5}u^{5/2} - \frac{2u^{3/2}}{3}\right]_4^{13}$

$= \frac{4}{135}\left(221\sqrt{13} - 18\right)$

b) Find the coordinate of the point where this tangent meets the curve again.

[4 marks]

When tangent meets the curve

$$t^3 - 2t = -\frac{1}{2}(t^2+1) + 2$$

$$\Rightarrow t = -1 \text{ or } t = 3/2$$

when $t = 3/2$

$$x = 13/4$$

$$y = 3/8$$

10 The number of bacteria in a petri dish is modelled mathematically by saying that the number of bacteria, n, increases at a rate proportional to the number of bacteria present at a given time. Suppose that the number of bacteria at 9 am is found to be 125. At 10 am there are 320 bacteria. At what time will there first be more than 2000 bacteria.

[6 marks]

$$\frac{dn}{dt} \propto n \Rightarrow \frac{dn}{dt} = kn$$

$$\Rightarrow n = n_0 e^{kt}$$

when $t = 0$, $n_0 = 125$

so $n = 125e^{kt}$

when $t = 60$, $n = 320$ (time in mins)

$320 = 125e^{k60}$

$\Rightarrow k = 0.015666787 64$

≈ 0.0157

so

$n = 125e^{0.0157t}$

when $n = 2000$, $t = 176.59 \approx 2$ hrs 57 minutes

so at 11:57.

11 Find the cartesian equation for the curve with parametric equations $x = \sin(\theta)$, $y = 3\sin(2\theta)$.

[3 marks]

$y = 3\sin 2\theta$

$= 6\sin\theta\cos\theta$

$\Rightarrow y^2 = 6\sin^2\theta\cos^2\theta$

$= 6x^2(1 - x^2)$

12 The vertical displacement of a particle attached to the end of a spring is given by the equation $y = A\cos\left(\frac{2t}{10}\right) + B\sin\left(\frac{2t}{10}\right)$.

a) Given that at time $t = 0$, $y(0) = 1.2$ and $\frac{\mathrm{d}y}{\mathrm{d}\cancel{x}t}(0) = 0.5$ find the values of the constants A and B.

[3 marks]

$t = 0,\ \cancel{y}y = 1.2 \Rightarrow A = 1.2$

$t - 0,\ \frac{dy}{dt} = \frac{1}{2} \Rightarrow B = 2.5$

Hence

$$y = 1.2\cos\left(\frac{2t}{10}\right) + 2.5\sin\left(\frac{2t}{10}\right)$$

b) Hence find the maximum displacement of the particle from the equilibrium point.

[4 marks]

$$1.2\cos\left(\frac{2t}{10}\right) + 2.5\sin\left(\frac{2t}{10}\right) = R\cos\left(\frac{2t}{10} - \alpha\right)$$

$$= R\left(\cos\left(\frac{2t}{10}\right)\cos(\alpha) + \sin\left(\frac{2t}{10}\right)\sin(\alpha)\right)$$

$\Rightarrow \cancel{\alpha}\ 1.2 = R\cos\alpha$

$2.5 = R\sin\alpha$

$\Rightarrow R = \sqrt{1.2^2 + 2.5^2}$

$= \frac{\sqrt{769}}{10}$

≈ 2.77

13 Consider the parallelogram $OACB$ (labelled anticlockwise). Let $\overrightarrow{OA} = \mathbb{a}$ and $\overrightarrow{OB} = \mathbb{b}$. Further, let M be a point that lies $\frac{1}{3}$ of the way along OA and N be a point that lies $\frac{2}{3}$ of the way along AC.

a) Find the vector $\overrightarrow{OC}$.

[1 mark]

$\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$

b) Find the vector $\overrightarrow{MN}$.

[3 marks]

$\overrightarrow{OM} = \frac{1}{3}\underset{\sim}{a}$

$\overrightarrow{AN} = \frac{2}{3}\underset{\sim}{b}$

$\overrightarrow{ON} = \underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$

so $\overrightarrow{MN} = \frac{2}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b} = \frac{2}{3}(\underset{\sim}{a} + \underset{\sim}{b})$

c) What can be deduced about the vectors $\overrightarrow{OC}$ and $\overrightarrow{MN}$ from parts (a) and (b)? And why?

[2 marks]

$\overrightarrow{OC}$ and $\overrightarrow{MN}$ are parallel since $\overrightarrow{MN}$ is a scalar multiple of $\overrightarrow{OC}$

14 Consider the equation $e^x \sin(x-3)+1=0$.

a) Show that there is a root between 0 and 2 for the above equation.

[2 marks]

Let $f(x) = e^x \sin(x-3)+1$

$f(0) \approx 0.8589$

$f(2) \approx -5.2177$

Sign change so root lies in the interval $[0,2]$

b) Use the Newton-Raphson process with $x_0 = 0.5$ to find the approximation x_1 to one of the roots of the equation above.

[3 marks]

$f'(x) = e^x(\sin(x-3)+\cos(x-3))$

Hence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{e^{x_n}(\sin(x_n-3))+1}{e^{x_n}(\sin(x_n-3)+\cos(x_n-3))}$$

$x_0 = \frac{1}{2}$

$x_1 = 0.505376628$

c) Using your calculator to find a reference value for this root, calculate the percentage error made in using one step of the Newton-Raphson iteration.

[2 marks]

Using the solve function gives a root of 0.5057388023.

so % error $\approx 3.72 \times 10^{-3}$%

15 Find, in exact form $\tan(75°)$.

[6 marks]

$$\sin(75) = \sin(45+30)$$
$$= \sin(45)\cos(30) + \cos(45)\sin(30)$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos(75) = \cos(45+30)$$
$$= \cos(45)\cos(30) - \sin(45)\sin(30)$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

So $$\tan(75) = \frac{\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \cancel{2} \; \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= 2+\sqrt{3}$$

16 Find the binomial expansion of $\dfrac{3x+5}{(1-x)(3x+1)}$ and state the range of values for which this expansion is valid.

[9 marks]

$$\frac{3x+5}{(1-x)(3x+1)} = \frac{3}{1+3x} + \frac{2}{1-x}$$

$$(1-x)^{-1} \approx 1 + x + x^2 + x^3 + x^4$$

$$(3x+1)^{-1} \approx 1 - 3x + 9x^2 - 27x^3 + 81x^4$$

Hence

$$\frac{3x+5}{(1-x)(3x+1)} \approx 3(1 - 3x + 9x^2 - 27x^3) + 2(1 + x + x^2 + x^3)$$

$$= 5 - 7x + 29x^2 - 79x^3$$

17 Find the area between the functions $f(x) = \cos(x)$ and $g(x) = \sin(x)$ in the interval $\left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$.

[4 marks]

$$\int_{-\frac{3\pi}{4}}^{\pi/4} (\cos(x) - \sin(x))\, dx = \left[\sin(x) + \cos(x)\right]_{-3\pi/4}^{\pi/4}$$

$$= 2\sqrt{2}$$

18 It is given that $\theta = \arcsin(2x)$.

a) Writing $\sin(\theta) = 2x$, use implicit differentiation to show that

$$\frac{d\theta}{dx} = \frac{A}{\sqrt{1 - Bx^2}}$$ where A and B are integers to be found.

[3 marks]

$$\sin\theta = 2x$$

$$\Rightarrow \cos\theta \frac{d\theta}{dx} = 2$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{2}{\cos\theta}$$

$$= \frac{2}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

b) Hence find the general solution of the differential equation

$$(1 - 4x^2)^{\frac{1}{2}}\frac{dy}{dx} = 2y$$

[5 marks]

$$\Rightarrow \int \frac{1}{y}\, dy = \int \frac{2}{\sqrt{1-4x^2}}\, dx$$

$$\Rightarrow \ln(y) = \arcsin(2x) + C$$

$$\Rightarrow y = e^{\arcsin(2x)} + k$$