



AQA FM Paper 2 2019 “Predicted Content”

Name:

Class:

Mark: / 124

1 Calculate $\sum_{r=5}^{10} r(r + 1)$

400

420

440

532

[1 mark]

2 The vectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ a \\ 1 \end{pmatrix}$ are perpendicular. Find the value of a .

3

-3

9

-9

[1 mark]

3 Let $z_1 = 2e^{i\frac{\pi}{3}}$ and $z_2 = z_2 = 3e^{i\frac{\pi}{12}}$. , if $-\pi < \arg(z) \leq \pi$, $\arg(z_1^3 z_2)$ is equal to

$-\frac{11\pi}{12}$

$\frac{13\pi}{12}$

$\frac{11\pi}{12}$

$\frac{7\pi}{36}$

[1 mark]

4 Show, with full reasoning, that $\int_{-\infty}^{\infty} xe^{-x^2} dx = 0$.

[5 marks]

$$\text{Consider } I_1 = \int_{-\infty}^0 xe^{-x^2} dx$$

$$= \lim_{t \rightarrow -\infty} \int_t^0 xe^{-x^2} dx$$

$$= \lim_{t \rightarrow -\infty} \left(-\frac{1}{2} e^{-x^2} \right) \Big|_t^0$$

$$= \lim_{t \rightarrow -\infty} \left(-\frac{1}{2} + \frac{1}{2} e^{-t^2} \right)$$

$$= -\frac{1}{2}$$

$$\text{Consider } I_2 = \int_0^{\infty} xe^{-x^2} dx$$

$$= \lim_{t \rightarrow \infty} \int_0^t xe^{-x^2} dx$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} e^{-x^2} \right) \Big|_0^t$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} e^{-t^2} + \frac{1}{2} \right)$$

$$= \frac{1}{2}$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} xe^{-x^2} dx &= I_1 + I_2 \\ &= 0 \end{aligned}$$

5 a) Show that

$$\frac{1}{k+2} - \frac{1}{k+4} = \frac{2}{k^2 + 6k + 8}$$

[2 marks]

$$\begin{aligned}\frac{1}{k+2} - \frac{1}{k+4} &= \frac{k+4 - (k+2)}{k^2 + 6k + 8} \\ &= \frac{2}{k^2 + 6k + 8}\end{aligned}$$

b) Hence, find an expression for

$$\sum_{k=1}^n \frac{2}{k^2 + 6k + 8}$$

[5 marks]

$$\sum_{k=1}^n \frac{2}{k^2 + 6k + 8} = \sum_{k=1}^n \frac{1}{k+2} - \frac{1}{k+4}$$

$$k=1 \quad \frac{1}{3} - \cancel{\frac{1}{5}}$$

$$k=2 \quad \frac{1}{4} - \cancel{\frac{1}{6}}$$

$$k=3 \quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$$

$$k=4 \quad \cancel{\frac{1}{6}} - \frac{1}{8}$$

$$k=5 \quad \cancel{\frac{1}{7}} - \frac{1}{9}$$

⋮

$$k=n-4 \quad \frac{1}{n-2} - \cancel{\frac{1}{n}}$$

$$k=n-3 \quad \frac{1}{n-1} - \frac{1}{n+1}$$

$$k=n-2 \quad \frac{1}{k} - \frac{1}{n+2}$$

$$k=n-1 \quad \frac{1}{k+1} - \frac{1}{n+3}$$

$$k=n \quad \frac{1}{n+2} - \frac{1}{n+4}$$

Hence

$$\sum \frac{2}{k^2 + 6k + 8} = \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7}{12} - \frac{1}{n+3} - \frac{1}{n+4}$$

- c) Using your result to part (b), or otherwise, evaluate

$$\sum_{k=1}^{\infty} \frac{2}{k^2 + 6k + 8}$$

[2 marks]

As $n \rightarrow \infty$

$$\sum_{k=1}^{\infty} \frac{2}{k^2 + 6k + 8} \rightarrow \frac{7}{12}$$

- 6 The circle $x^2 + y^2 = 1$ undergoes three transformations. It is translated by the vector $\begin{pmatrix} -\frac{2}{5} \\ \frac{3}{4} \end{pmatrix}$ before being stretched horizontally with a scale factor 5. Finally it is stretched vertically with a scale factor 4.

- a) Find an equation for the resulting conic and determine its nature.

[5 marks]

Translation with vector $\begin{pmatrix} -\frac{2}{5} \\ \frac{3}{4} \end{pmatrix}$

$$\text{so } \left(x + \frac{2}{5}\right)^2 + \left(y - \frac{3}{4}\right)^2 = 1$$

Horizontal stretch scale factor 5

$$\text{so } \left(\frac{x}{5} + \frac{2}{5}\right)^2 + \left(y - \frac{3}{4}\right)^2 = 1$$

Vertical sketch with scale factor 4

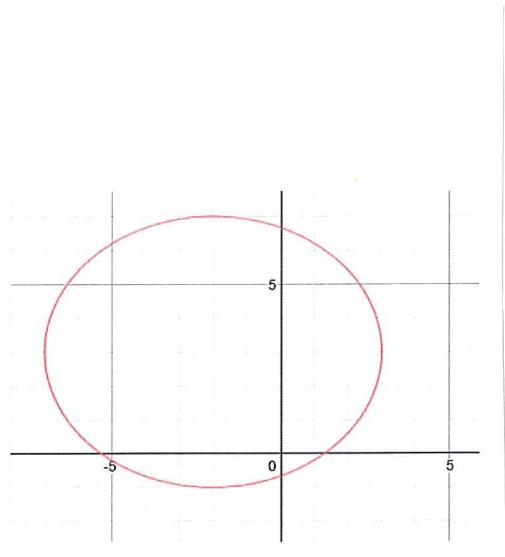
$$\text{so } \left(\frac{x+2}{5}\right)^2 + \left(\frac{y-3}{4}\right)^2 = 1$$

$$\Rightarrow \frac{(x+2)^2}{25} + \frac{(y-3)^2}{16} = 1$$

Ellipse centre $(-2, 3)$

- b) Sketch the resulting conic.

[2 marks]



- 7 a) Prove that $\text{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Let $y = \text{arsinh}(x)$

[4 marks]

$$\Rightarrow \sinh(y) = x$$

$$\Rightarrow \frac{e^y - e^{-y}}{2} = x$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$
$$= x \pm \sqrt{x^2 + 1}$$

$$\text{but } e^y > 0 \Rightarrow e^y = x + \sqrt{x^2 + 1}$$

$$\text{Hence } y = \ln(x + \sqrt{x^2 + 1})$$

- b) Hence, find in logarithmic form the solutions to
 $\cosh^2(x) - 2 \sinh(x) - 4 = 0$

[4 marks]

$$\cosh^2(x) - 2 \sinh(x) - 4 = 0$$

$$\Rightarrow \cosh^2(x) - 1 - 2 \sinh(x) - 3 = 0$$

$$\Rightarrow \sinh^2(x) - 2 \sinh(x) - 3 = 0$$

$$\Rightarrow (\sinh(x) + 1)(\sinh(x) - 3) = 0$$

$$\text{so } \sinh(x) = -1 \text{ or } \sinh(x) = 3$$

$$\therefore x = \ln(-1 + \sqrt{1+1}) \text{ or } x = \ln(3 + \sqrt{10})$$
$$= \ln(\sqrt{2} - 1)$$

8 a) Find a reduction formula for $\int_0^{\frac{\pi}{2}} \cos^n(x) dx$

[4 marks]

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n(x) dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1}(x) \cos(x) dx$$

$$\text{Let } u = \cos^{n-1}(x)$$

$$\frac{du}{dx} = -(n-1) \cos^{n-2}(x) \sin(x)$$

$$\frac{du}{dx} = \cos(u)$$

$$v = \sin(x)$$

Hence,

$$I_n = \left[\sin(x) + \cos^{n-1}(x) \right]_0^{\frac{\pi}{2}} + \int \cos^{n-2}(x) \sin^2(x) dx^{(n-1)}$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2}(x) (1 - \cos^2(x)) dx$$

$$= (n-1) \left[\int_0^{\frac{\pi}{2}} \cos^{n-2}(x) dx - \int_0^{\frac{\pi}{2}} \cos^n(x) dx \right]$$

\Rightarrow

$$I_n = (n-1)(I_{n-2} - I_n)$$

$$\Rightarrow I_n (1+n-1) = (n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

- b) Hence, evaluate $\int_0^{\frac{\pi}{2}} \cos^2(x)(1 + \cos^2(x)) dx.$

[4 marks]

$$\begin{aligned}
 & \int_0^{\frac{\pi}{2}} \cos^2(x)(1 + \cos^2(x)) dx \\
 &= \int_0^{\frac{\pi}{2}} \cos^2(x) dx + \int_0^{\frac{\pi}{2}} \cos^4(x) dx \\
 &= I_2 + I_4 \\
 &= I_2 + \frac{3}{4} I_2 \\
 &= \frac{7}{4} I_2 \\
 &= \frac{7}{4} \cdot \frac{1}{2} I_0 \\
 &= \frac{7}{8} \int_0^{\frac{\pi}{4}} \cos^2(x) dx \\
 &= \frac{7\pi}{16}
 \end{aligned}$$

- 9 Consider the function $y = \frac{x^2+2}{x}.$

- a) Find the equations of the asymptotes to this curve.

[3 marks]

Vertical asymptote at $x = 0$

$$\begin{aligned}
 & x \sqrt{x^2+2} \quad \text{so oblique asymptote at} \\
 & \frac{x^2}{2} \quad y = x
 \end{aligned}$$

- b) Find, using quadratic theory, the stationary points of the curve
 $y = \frac{x^2 + 2}{x}$. Solutions employing the calculus will not be awarded marks.

[5 marks]

Let

$$k = \frac{x^2 + 2}{x}$$

$$\Rightarrow kx = x^2 + 2$$

$$\Rightarrow 0 = x^2 - kx + 2$$

Discriminant, $b^2 - 4ac = 0$

$$k^2 - 4 \cdot 1 \cdot 2 = 0$$

$$\Rightarrow k^2 = 8$$

$$\Rightarrow k = \pm\sqrt{8} \\ = \pm 2\sqrt{2}$$

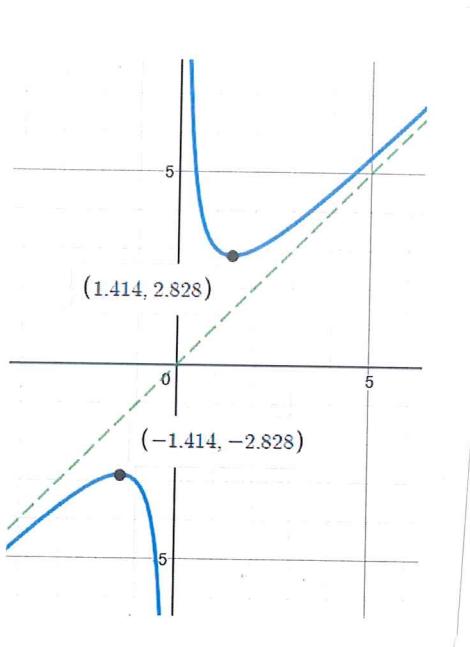
When $y = 2\sqrt{2}$, $x = \sqrt{2}$

when $y = -2\sqrt{2}$, $x = -\sqrt{2}$

So turning points at $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$

c) Hence, sketch the curve $y = \frac{x^2 + 2}{x}$.

[3 marks]



10 a) Solve, analytically, the differential equation

$$\frac{dy}{dx} = x + y$$

where $y(1) = 4$.

[5 marks]

$$\frac{dy}{dx} - y = x$$

I.F. $e^{-\int dx} = e^{-x}$

so $e^{-x} y(x) = e^{-x}(-x - 1) + C$

$$\text{Ans } y(0) = 4$$

$$y(x) = 6e^{x-1} - 5x - 1$$

- b) Use Euler's method with a step size of 0.1 to find an approximate value for $y(1.3)$.

[3 marks]

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$n=0 \quad x=1, y=4$$

$$n=1 \quad x=1.1, y = 4 + 0.1 \times (1+4) \\ = 4.5$$

$$n=2 \quad x=1.2, y = 4.5 + 0.1 \times (1.1+4.5) \\ = 5.06$$

$$n=3 \quad x=1.3, y = 5.06 + 0.1 \times (1.2+5.06) \\ = 5.686$$

$$\text{So } y(1.3) \approx 5.686$$

- c) Consider the Improved Euler Method shown below

$$y_{r+1} = y_r + 2hf(x_r, y_r), \quad x_{r+1} = x_r + h$$

Using the boundary value $y(1) = 4$ and the value calculated in part (b) for $y(1.1)$ find an approximate value for $y(1.3)$ using this Improved Euler Method.

[3 marks]

$$r=0, x=1, y=4$$

$$r=1, x=1.1, y=4.5$$

$$r=2, x=1.2, y = 4 + 2 \times 0.1 \times (1.1 + 4.5) \\ = 5.12$$

$$r=3, x=1.3, y = 4.5 + 2 \times 0.1 \times (1.2 + 5.12) \\ = 5.764$$

So $y(1.3) \approx 5.764$

- d) With reference to the exact solution found in part (a) comment on the accuracy of the Euler Method and the Improved Euler Method.

Using the solution found in (a) $y(1.3) = 5.799152845$
 As expected the Improved Euler is more accurate than the Euler method.

[2 marks]

11 Find

$$I = \int \frac{4x}{(x+1)(x^2+4)} dx$$

[8 marks]

Consider

$$\frac{x}{(x+1)(x^2+4)} = \frac{x+4}{5(x^2+4)} - \frac{1}{5(x+1)}$$

Hence

$$\begin{aligned} I &= \int \frac{4x}{(x+1)(x^2+4)} \\ &= \frac{4}{5} \int \left(\frac{x}{x^2+4} + \frac{4}{x^2+4} - \frac{1}{x+1} \right) dx \\ &= \frac{2}{5} \ln|x^2+4| - \frac{4}{5} \ln|x+1| + \frac{8}{5} \arctan\left(\frac{x}{2}\right) \end{aligned}$$

12 Evaluate $\lim_{x \rightarrow 0} \frac{\arcsin(2x)}{\arctan(7x)}$.

[5 marks]

Since $\lim_{x \rightarrow 0} \arcsin(2x) = 0$ and $\lim_{x \rightarrow 0} \arctan(7x) = 0$

we can apply L'Hôpital's rule

$$\lim_{x \rightarrow 0} \frac{\arcsin(2x)}{\arctan(7x)} = \lim_{x \rightarrow 0} \left(\frac{2 \cdot \frac{1}{\sqrt{1-(2x)^2}}}{7} \right)$$

$$= \lim_{x \rightarrow 0} \frac{2}{\sqrt{1-4x^2}} \cdot \frac{1+49x^2}{7}$$

$$= \lim_{x \rightarrow 0} \frac{2}{7} \left(\frac{1+49x^2}{\sqrt{1+4x^2}} \right)$$

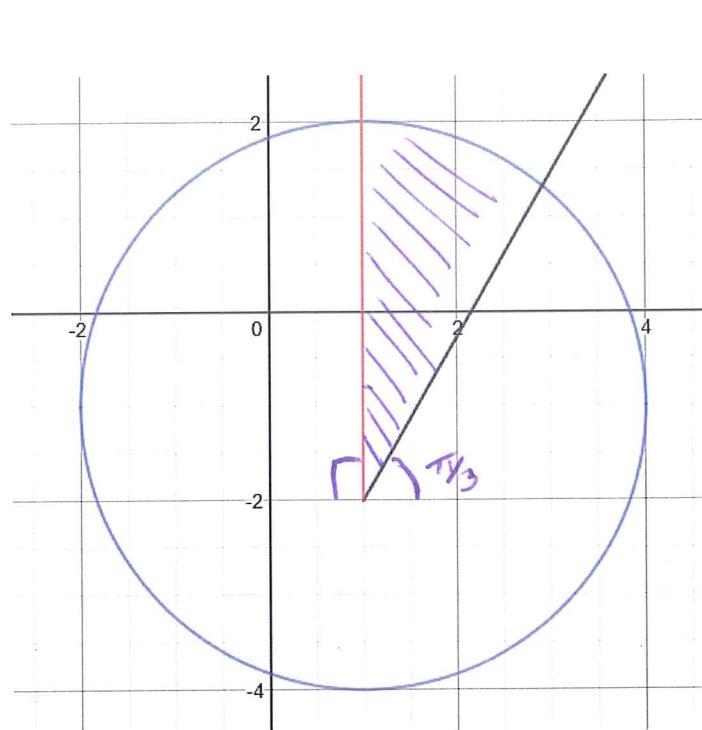
$$= \frac{2}{7}$$

13 On an Argand diagram shade the region given by

$$\{z : |z - 1 + i| < 3\} \cap \left\{ \frac{\pi}{3} \leq \arg(z - 1 + 2i) \leq \frac{\pi}{2} \right\}$$

$\Rightarrow \arg(z - 1 + 2i) < \frac{\pi}{2}$

[5 marks]



14 Prove, by induction, that $3^{2n} + 7$ is divisible by 8. $\forall n \in \mathbb{N}$

[7 marks]

Step 1: When $n=1$,

$$\text{LHS} = 3^{2 \times 1} + 7$$

$$= 9 + 7$$

$$= 16 = 2 \times 8 \text{ so divisible by 8.}$$

Step 2: We assume the result holds for $n=k$, i.e.

$$8 \mid 3^{2k} + 7$$

Step 3: When $n=k+1$

$$3^{2(k+1)} + 7 = 3^{2k+2} + 7$$

$$= 3^2(3^{2k}) + 7$$

$$= 9(3^{2k} + 7) - 63 + 7$$

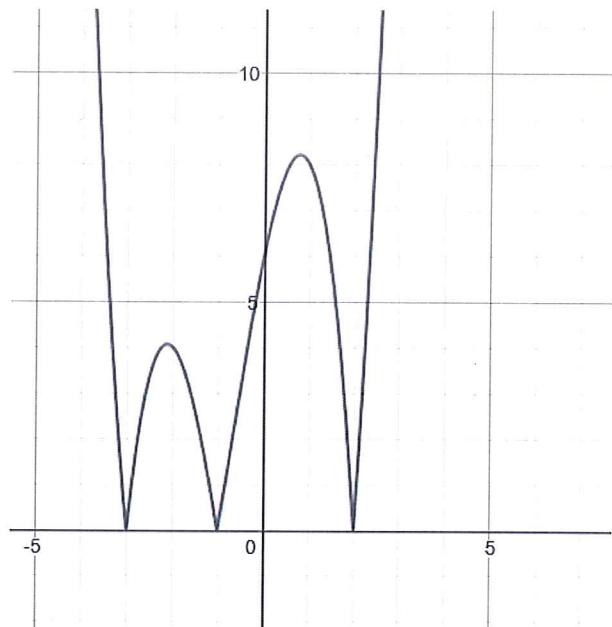
$$= 9(3^{2k} + 7) - \underbrace{56}_{\substack{\text{divisible by 8} \\ \text{by inductive} \\ \text{hypothesis}}}$$

$$56 = 7 \times 8$$

Step 4: Since $3^{2n} + 7$ is divisible by 8 for $n=1$, and we have shown that if the result holds for $n=k$ then it also holds for $n=k+1$. Hence, by the principle of mathematical induction the result holds for all $n \in \mathbb{N}$.

15 Sketch $f(x) = |(x+1)(x-2)(x+3)|$

[6 marks]



16 a) Find the invariant lines under the transformation

$$T = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

[6 marks]

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} -\frac{3}{5}x + \frac{4}{5}(mx + c) \\ \frac{4}{5}x + \frac{3}{5}(mx + c) \end{pmatrix}$$

$$\text{but } y' = mx' + c$$

$$\stackrel{so}{\frac{4}{5}x + \frac{3}{5}(mx + c) = m\left(-\frac{3}{5}x + \frac{4}{5}(mx + c)\right) + c}$$

$$\Rightarrow \frac{4}{5}x + \frac{3}{5}mx + \frac{3}{5}c = -\frac{3mx}{5} + \frac{4m^2x}{5} + \frac{4mc}{5} + c$$

$$\Rightarrow 0 = x\left(\frac{4m^2 - 6m - 4}{5}\right) + c\left(\frac{4m + 2}{5}\right)$$

$$\Rightarrow 0 = (4m^2 - 6m - 4)c + c(4m + 2)$$

Must be true for all x

$$\Rightarrow 0 = 2(m-2)(2m+1)c + 2(2m+1)c$$

If $m = -\frac{1}{2}$ then c can be anything

If $m=2$ then $c=0$

Hence $y=2x$
and $y=-\frac{1}{2}x+c$ are the invariant lines

- b) Describe the transformation T geometrically.

[2 marks]

Reflection in the line $y=2x$

- 17 a) Compute the arc length of $y = \ln(\sec(x))$ between $x = 0$ and $x = \frac{\pi}{3}$.

[5 marks]

$$S = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y = \ln(\sec(x)) \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{\sec(x)} \cdot \sec(x)\tan(x) = \tan(x)$$

Hence,

$$S = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2(x)} dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2(x)} dx$$

$$= \int_0^{\frac{\pi}{3}} \sec(x) dx$$

$$= \left[\ln|\sec(x) + \tan(x)| \right]_0^{\frac{\pi}{3}}$$

$$= \ln|2 + \sqrt{3}|$$

- b) The curve described in part (a), when rotated around the x -axis forms the curved boundary of titanium paperweight that has been machined on a CNC lathe. Showing full working, approximately determine the volume of the paperweight by employing the mid-ordinate rule where the number of strips, n , is 3.

[4 marks]

$$\text{Volume} = \pi \int y^2 dx \text{ with } y^2 = (\ln(\text{secl})^2) dx$$

Ordinates	Midpoint	y -value
$x=0$	$\pi/18$	$2.343603209 \times 10^{-4}$ y_1
$x=\frac{\pi}{9}$	$\pi/6$	0.0206902437 y_2
$x=\frac{2\pi}{9}$	$5\pi/18$	0.1953117785 y_3
$x=\frac{\pi}{3}$		

$$\text{Hence, Volume} \approx \pi \times \frac{\pi}{9} \times (y_1 + y_2 + y_3)$$

$$\approx 0.237129727$$

18 Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$

a) Find the eigenvalues of A .

[3 marks]

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(5-\lambda) - 1 \times 3 = 0$$

$$5 - 5\lambda - \lambda + \lambda^2 + 3 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 4) = 0$$

Hence eigenvalues for A are $\lambda_1 = 2$
 $\lambda_2 = 4$

b) Find the corresponding eigenvectors.

[4 marks]

Eigenvectors satisfy $A\tilde{v} = \lambda\tilde{v}$ for λ an eigenvalue

Hence,

$$\underline{\lambda_1 = 2}: \quad \tilde{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\underline{\lambda_2 = 4}: \quad \tilde{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- c) Find the inverse matrix A^{-1} and verify that $\lambda_1 = \frac{1}{2}$ is an eigenvalue of A^{-1} with corresponding eigenvector $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

[3 marks]

$$A^{-1} = \frac{1}{8} \begin{pmatrix} 5 & -3 \\ 1 & 1 \end{pmatrix}$$

Since $|A| = 1 \times 5 - 3 \times -1$

$$= 8$$

- d) Explain why 16 must be an eigenvector of A^4 .

[2 marks]

For (λ, v) to be an eigenvalue - eigenvector pair

$$Av = \lambda v$$

$$A^4 v = A^3 Av$$

$$= A^3 \lambda v$$

$$= A^2 A \lambda v$$

$$= \lambda A^2 A v$$

$$= \lambda A^3 \lambda v$$

$$= \lambda^2 A^3 v$$

$$= \dots$$

$$= \lambda^4 v$$

As 2 is an eigenvalue of A , $2^4 = 16$ must be an eigenvalue of A^4