Getting Better at Setting out Mathematics

We will use some properties of the real numbers for a few of these questions. They are summarised below, where x, y and z are real numbers.

Property Name	Property
Identity Properties	$x + 0 = x$ and $x \cdot 1 = x$.
Inverse Properties	$x + (-x) = 0$ and $x \cdot \frac{1}{x} = 1$.
Commutative Properties	$x + y = y + x$ and $xy = yx$.
Associative Properties	$(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.
Distributive Properties	$x(y + z) = xy + xz$ and $(y + z)x = yx + zx$

The closure properties of the different number systems are also sometimes required.

Number System	Closed Under
$\mathbb{N}$, the natural numbers.	Addition and multiplication.
$\mathbb{Z}$, the integers.	Addition, subtraction and multiplication.
$\mathbb{Q}$, the rational numbers.	Addition, subtraction, multiplication, and division by nonzero rational numbers.
$\mathbb{R}$, the real numbers.	Addition, subtraction, multiplication and division by nonzero real numbers.

Some General Guidelines:

- Start with a statement of what we are trying to prove. Then miss a line, write "Proof:" and begin the proof.
- Begin the proof by clearly stating any assumptions that can be made (These are often clear from the statement of what is to be proved. For example, suppose we are asked to prove "Prove that for any odd integers, $x, y,$ the sum $x + y$ is even" the assumptions would be "We assume that x and y are both odd integers".
- Clearly display any important manipulations and algebraic processes should be displayed and not contained within sentences.
- Use the pronoun "we" as opposed to "I".

<u>Example of Using a "Know - Why" Table</u>

Suppose we wish to prove that the sum of two odd integers is even then the first step is to write our proposition mathematically.

Proposition: If x and y are odd integers then $x + y$ is an even integer.

We can use a "Know-Why" table to organise our thinking.
We now identify our hypotheses / assumptions and the conclusion:

H: x and y are odd integers.
C: $x + y$ is an even integer.

These statements can then be entered into a "Know-Why?" table.

Step	Know	Why
H	x and y are odd integers.	Assumption.
H1		
$\vdots$		
C1		
C	$x + y$ is an even integer.	

We now need to consider how to fill in the gaps between H and C. It is often instructive to ask ourselves "How can we conclude that C is true?". In this case, we could conclude that C is true if we show that there exists an integer p such that $x + y = 2p + 1$, This then forms our statement for C1:

Step	Know	Why
H	x and y are odd integers.	Assumption.
H1		
$\vdots$		
C1	There exists an integer p such that $x + y = 2q$	
C	$x + y$ is an even integer.	Definition of an even integer.

Note that often there will be multiple options for C1, and that choosing early on could result in some time being spent unable to fill in the missing gaps and then having to change our C1. This isn't time wasted however, often we learn valuable insights about what we are trying to prove by making these attempts.

We now need to complete the table.

Step	Know	Why
H	x and y are odd integers.	Assumption.
H1	There exist integers m, n such that $x = 2m + 1$ and $y = 2n + 1$	By the definition of an odd number.
H2	$x + y = (2m + 1) + (2n + 1)$	By substitution
H3	$x + y = 2m + 2n + 2$	By algebraic manipulation.
H4	$x + y = 2(m + n + 1)$	By algebraic manipulation.
H5	$(m + n + 1)$ is an integer	By the closure properties of $\mathbb{Z}$.
C1	There exists an integer p such that $x + y = 2p$.	Let $p = m + n + 1$.
C	$x + y$ is an even integer.	Definition of an even integer.

We now have the steps for a complete proof listed, however proofs are not presented in such a table and so we need to write it this proof in a mathematically acceptable way.

Proposition 1: If x and y are odd integers then $x + y$ is an even integer.

Proof: We assume that x and y are odd integers and seek to prove that $x + y$ is an even integer. Since x and y are both odd, there exist integers m and n such that

$$x = 2m + 1 \text{ and } y = 2n + 1$$

With this we find,

$$\begin{aligned} x + y &= (2m + 1) + (2n + 1), \\ &= 2m + 2n + 2, \\ &= 2(m + n + 1) \end{aligned}$$

Since $\mathbb{Z}$ is closed under addition, $(m + n + 1)$ is an integer and we note that $x + y$ is of the form $x + y = 2p$ where $p = m + n + 1$. Thus, we can conclude, by the definition, that $x + y$ is an even integer.

Question 1:

Use a "Know-Why" table to structure a proof to show that if we multiply two odd integers then the result is absolutely odd.

Proposition:

H:

C:

Question 2:

Prove that, if x is an odd integer then $7x + 5$ is an even integer.

Proposition:

H:

C:

Question 3:

Prove that the product of two odd integers is odd.

Proposition:

H:

C: