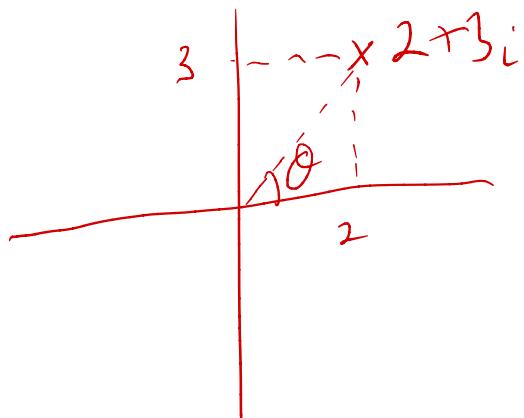




### Do Now

- 1) Find the exponential form and modulus argument form of the complex number  $2 + 3i$ .



$$|2+3i| = \sqrt{4+9} = \sqrt{13}$$

$$\theta = \arctan\left(\frac{3}{2}\right)$$

$$\approx 0.98^\circ$$

$$\begin{aligned} \text{so } z &= 2+3i \\ &= \sqrt{13} e^{0.98i} \\ &= \sqrt{13} (\cos(0.98^\circ) + i \sin(0.98^\circ)) \end{aligned}$$

2) Find  $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2$

$$= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{4} - \frac{i\sqrt{3}}{2} - \frac{3}{4}$$

$$= \frac{-1}{2} - i\sqrt{3}$$

3) Find the square root of  $-28 + 96i$

Let  $z = a+ib$ , then  $z^2 = -28 + 96i$ , where  $a, b \in \mathbb{R}$   
then

$$(a+ib)^2 = -28 + 96i$$

$$a^2 + 2abi - b^2 = -28 + 96i$$

Re]  $a^2 - b^2 = -28 \quad ①$

Im]  $2ab = 96 \quad ②$

$$\Rightarrow b = \frac{96}{2a} = \frac{48}{a}$$

Substitute  $b = \frac{48}{a}$  into ①

$$a^2 - \frac{48^2}{a^2} = -28$$

$$\Rightarrow a^4 + 28a^2 - 2304 = 0$$

$$\Rightarrow (a^2 - 36)(a^2 + 64).$$

Since  $a \in \mathbb{R}$ ,  $a^2 + 64 = 0$  has no solutions. So

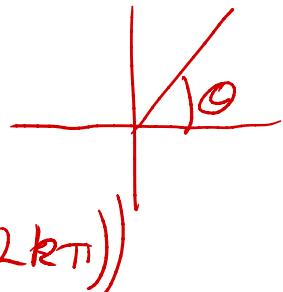
$$a = \pm 6.$$

$$\left. \begin{array}{l} \text{When } a = 6, b = 8 \\ \text{When } a = -6, b = -8 \end{array} \right\} \text{ so } \sqrt{-28+96i} = 6+8i \text{ or } -6-8i$$

**n<sup>th</sup> Roots of a Complex Number**

Let  $z$  be a complex number and  $w$  also be a complex number. Then if  $n$  is a positive integer the equation  $z^n = w$  has  $n$  distinct solutions.

For the examples we are going to make use of the fact that



$$z = r(\cos(\theta) + i \sin(\theta))$$

$$= r(\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi))$$

**Example**

a) Solve the equation  $z^3 = 1$   $\Rightarrow z^3 = \cos(0) + i \sin(0)$

$$(r(\cos(\theta) + i \sin(\theta)))^3 = \cos(0) + i \sin(0)$$

$$= \cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)$$

Applying De Moivre's Thm to LHS

$$r^3(\cos(3\theta) + i \sin(3\theta)) = \cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi),$$

$k \in \mathbb{Z}$

$$r^3 = 1 \Rightarrow r = 1$$

$$3\theta = \theta + 2k\pi \Rightarrow \theta = \frac{2k\pi}{3}$$

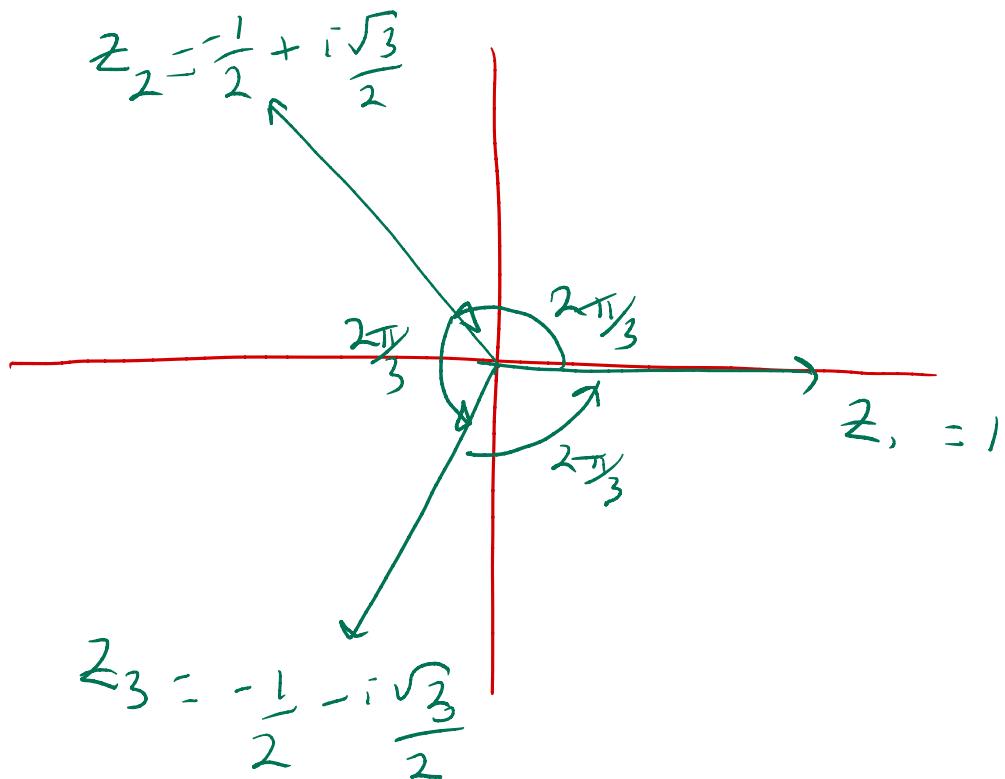
$k=0$   $\theta = 0 \Rightarrow z = \cos(0) + i \sin(0) = 1$

$k=1$   $\theta = \frac{2\pi}{3} \Rightarrow z = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k=-1$   $\theta = -\frac{2\pi}{3} \Rightarrow z = \cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right)$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

- b) Represent the solutions from (a) on an Argand diagram.



- c) Show that the cube roots of 1 can be written as  $1, \omega$  and  $\omega^2$  where  $1 + \omega + \omega^2 = 0$

$$\begin{aligned} \text{Let } \omega &= z_2 \\ &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ &= e^{\frac{2\pi i}{3}} \end{aligned}$$

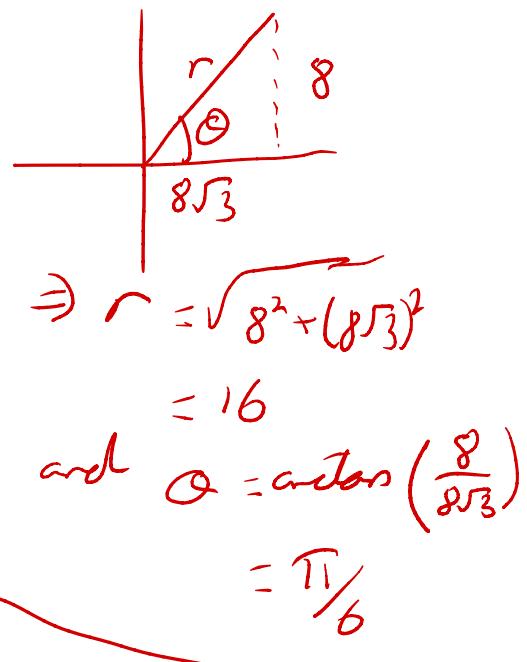
$$\begin{aligned} \Rightarrow \omega^2 &= e^{\frac{4\pi i}{3}} \\ &= e^{-\frac{2\pi i}{3}} \\ &= z_3 \end{aligned}$$

$$\begin{aligned} \text{Hence, } 1 + \omega + \omega^2 &= 1 + -\frac{1}{2} + i\frac{\sqrt{3}}{2} + -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

**Example**

Solve the equation  $z^4 = 8\sqrt{3} + 8i$

$$\begin{aligned}
 & (r(\cos(\theta) + i\sin(\theta)))^4 = 16 \left[ \cos\left(\frac{\pi}{6} + 2k\pi\right) \right. \\
 & \quad \left. + i\sin\left(\frac{\pi}{6} + 2k\pi\right) \right] \\
 \Rightarrow r^4(\cos(4\theta) + i\sin(4\theta)) &= 16 \left[ \cos\left(\frac{\pi}{6} + 2k\pi\right) + i\sin\left(\frac{\pi}{6} + 2k\pi\right) \right]
 \end{aligned}$$



$$r^4 = 16 \Rightarrow r = 2$$

$$4\theta = \frac{\pi}{6} + 2k\pi \Rightarrow \theta = \frac{\pi}{24} + \frac{k\pi}{4}$$

$$\underline{k=0}: \theta_1 = \frac{\pi}{24} \text{ and } z_1 = 2 \left( \cos\left(\frac{\pi}{24}\right) + i\sin\left(\frac{\pi}{24}\right) \right)$$

$$\underline{k=1} \quad \theta_2 = \frac{\pi}{24} + \frac{\pi}{2} = \frac{13\pi}{24} \text{ and } z_2 = 2 \left( \cos\left(\frac{13\pi}{24}\right) + i\sin\left(\frac{13\pi}{24}\right) \right)$$

$$\underline{k=-1} \quad \theta_3 = \frac{\pi}{24} - \frac{\pi}{2} = -\frac{11\pi}{24} \text{ and } z_3 = 2 \left( \cos\left(-\frac{11\pi}{24}\right) + i\sin\left(-\frac{11\pi}{24}\right) \right)$$

$$\underline{k=-2} \quad \theta_4 = \frac{\pi}{24} - \pi = -\frac{23\pi}{24} \text{ and } z_4 = 2 \left( \cos\left(-\frac{23\pi}{24}\right) + i\sin\left(-\frac{23\pi}{24}\right) \right)$$

**Exercise**

- 1) Find the fifth roots of unity.

$$\begin{aligned} \text{Let } z^5 &= 1 \\ &= 1(\cos(0) + i\sin(0)) \\ &= 1(\cos(2k\pi+0) + i\sin(2k\pi+0)) \end{aligned}$$

By De Moivre's Theorem

$$r^5 (\cos(5\theta) + i\sin(5\theta)) = \cos(2k\pi) + i\sin(2k\pi)$$

$$\exists r^5 = 1 \Rightarrow r = 1 \text{ and } \theta = \frac{2k\pi}{5}$$

$$k=0 \Rightarrow z_1 = 1$$

$$k=-1 \Rightarrow z_2 = \cos\left(\frac{-2\pi}{5}\right) + i\sin\left(\frac{-2\pi}{5}\right)$$

$$k=1 \Rightarrow z_3 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$$

$$k=-2 \Rightarrow z_4 = \cos\left(\frac{-4\pi}{5}\right) + i\sin\left(\frac{-4\pi}{5}\right)$$

$$k=2 \Rightarrow z_5 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

2) Solve  $z^3 - 27 = 0 \Rightarrow z^3 = 27$   
 $= 27(\cos(0+2k\pi) + i\sin(0+2k\pi))$

By De Moivre's Theorem

$$27(\cos(2k\pi) + i\sin(2k\pi)) = r^3(\cos(3\theta) + i\sin(3\theta))$$

$$\text{So } r^3 = 27 \Rightarrow r = 3$$

$$3\theta = 2k\pi \Rightarrow \theta = \frac{2k\pi}{3}$$

When  $k=0$ ,  $\theta = \frac{2k\pi}{3} = 0$

$$\text{so } z_1 = 3$$

When  $k=1$ ,  $\theta = \frac{2\pi}{3}$

$$= -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$\text{so } z_2 = 3 \left( \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \right)$$

When  $k=-1$ ,  $\theta = -\frac{2\pi}{3}$

$$\text{so } z_3 = 3 \left( \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) \right)$$

$$= -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

- 3) Solve the equation  $z^4 - \frac{75}{2} - \frac{81\sqrt{3}}{2}i = 3$ .

$$\begin{aligned} z^4 &= 3 + \frac{75}{2} + \frac{81\sqrt{3}}{2}i \quad \left| \begin{array}{l} r \\ \theta \end{array} \right. \\ &= \frac{81}{2} + \frac{81\sqrt{3}}{2}i \end{aligned}$$

$$\text{So } r = 81$$

$$\theta = \arctan\left(\frac{\frac{81\sqrt{3}}{2}}{\frac{81}{2}}\right) = \frac{\pi}{3}$$

So by De Moivre's Theorem

$$z^4 = r^4 (\cos(4\theta) + i \sin(4\theta)) = 81 \left( \cos\left(\frac{\pi}{3} + 2k\pi\right) + i \sin\left(\frac{\pi}{3} + 2k\pi\right) \right)$$

$$\text{So, } r^4 = 81 \Rightarrow r = 3$$

$$\text{and } 4\theta = \frac{\pi}{3} + 2k\pi \Rightarrow \theta = \frac{\pi}{12} + \frac{k\pi}{2}$$

When  $k=0$        $\theta = \frac{\pi}{12}$ , so  $z_1 = 3e^{\frac{i\pi}{12}}$

When  $k=-1$        $\theta = \frac{\pi}{12} - \frac{\pi}{2}$ , so  $z_2 = 3e^{-\frac{i5\pi}{12}}$

when  $k=1$        $\theta = \frac{\pi}{12} + \frac{\pi}{2}$ , so  $z_3 = 3e^{\frac{i7\pi}{12}}$

when  $k=2$        $\theta = \frac{\pi}{12} - \frac{\pi}{1}$ , so  $z_4 = 3e^{-\frac{i11\pi}{12}}$

- 4) Solve the equation  $1 + z + z^2 + z^3 + z^4 + z^5 = 0$ .

*Hint: The sum of a geometric series will be useful here.*

$$1 + z + z^2 + z^3 + z^4 + z^5 = 0 \quad \begin{array}{l} \text{geometric series,} \\ \nearrow \nearrow \nearrow \\ \times z \quad \times z \quad \times z \end{array}$$

*first term 1,  
ratio  $z$   
number of terms 6,*

Hence

$$1 + z + z^2 + z^3 + z^4 + z^5 = 0$$

$$\Rightarrow \frac{1 - z^6}{1 - z} = 0$$

$$\Rightarrow 1 - z^6 = 0$$

$$\Rightarrow z^6 = 1$$

So the solutions are the 6 roots of unity, excluding 1 since this wouldn't satisfy the original equation.

So roots are

$$z_1 = e^{\frac{2\pi i}{3}}$$

$$z_2 = e^{-\frac{2\pi i}{3}}$$

$$z_3 = e^{\frac{4\pi i}{3}}$$

$$z_4 = e^{-\frac{4\pi i}{3}}$$

$$z_5 = -1$$

