

Solutions

AQA A-Level Further Maths 2022 Paper 2

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100



- 1 Let $z = 3 - 2i$. Find $\frac{z^2}{z^*}$

$$3 + 2i$$

$$-\frac{1}{13}(9 + 46i)$$

$$\frac{1}{13}(-9 + 46i)$$

$$\frac{1}{13}(5 + 12i)$$

[1 mark]

- 2 Find the matrix representing an anticlockwise rotation by 30° centre the origin, followed by a reflection in the line $y = x$.

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

[1 mark]

rotation by 30° anticlockwise = $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

reflection in $y = x$ $\begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}$

so

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

- 3 Let M be the value of the mean value of the function $f(x)$ over the interval $[a, b]$. Then the mean value of the function $f(x) + d$ over the interval $[a, b]$ is equal to:

$$M$$

$$M + (b - a)$$

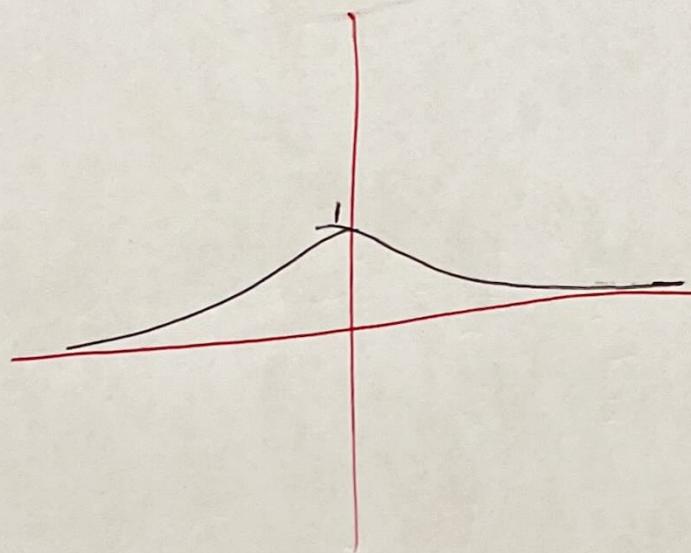
$$M + d$$

$$M + d(b - a)$$

[1 mark]

- 4 Sketch the function $y = \operatorname{sech}(x)$

[1 mark]



- 5 Prove by induction that the following statement is true for all $n \in \mathbb{N}$.

$$\sum_{r=1}^{2n} r^2(r+1) = \frac{1}{3}n(n+1)(2n+1)(6n+1)$$

Let $P(n)$ be the statement $\sum_{r=1}^{2n} r^2(r+1) = \frac{1}{3}n(n+1)(2n+1)(6n+1)$

[7 marks]

Step 1: When $n=1$

$$\text{LHS} = 1^2(1+1) + 2^2(2+1) \\ = 14$$

$$\text{RHS} = 14$$

Hence $P(1)$ is true.

Step 2: We assume $P(k)$ is true, i.e.

$$\sum_{r=1}^{2k} r^2(r+1) = \frac{1}{3}k(k+1)(2k+1)(6k+1)$$

Step 3: When $n=k+1$

$$\text{[RTJ]}: \sum_{r=1}^{2(k+1)} r^2(r+1) = \frac{1}{3}(k+1)(k+2)(2k+3)(6k+7)$$

$$\sum_{r=1}^{2(k+1)} r^2(r+1) = \sum_{r=1}^{2k} r^2(r+1) + (2k+1)^2(2k+2) \\ + (2k+2)^2(2k+3)$$

$$= \frac{1}{3}k(k+1)(2k+1)(6k+1) + (2k+1)^2(2k+2) + (2k+2)^2(2k+3)$$

$$\begin{aligned}
 &= \frac{1}{3} \left[k(k+1)(2k+1)(6k+1) + 12(k+1)(2k+1)^2 + 6(k+1)^2(2k+3) \right] \\
 &= \frac{1}{3} (k+1) \left[k(2k+1)(6k+1) + 12(2k+1)^2 + 6(k+1)(2k+3) \right] \\
 &= \frac{1}{3} (k+1) \left[12k^3 + 8k^2 + k + 48k^2 + 72 + 12k^2 + 30k + 18 \right] \\
 &= \frac{1}{3} (k+1) \left[k(2k+1)(6k+1) + \frac{6}{3} (k+1)(2k+1)^2 + \frac{12}{3} (k+1)^2(2k+3) \right] \\
 &= \frac{1}{3} (k+1) \left[k(2k+1)(6k+1) + 6(2k+1)^2 + 12(2k+3)(k+1) \right] \\
 &= \frac{1}{3} (k+1) \left[12k^3 + 8k^2 + k + 24k^2 + 24k + 6 + 24k^2 + 60k + 36 \right] \\
 &= \frac{1}{3} (k+1) (12k^3 + 56k^2 + 85k + 42) \\
 &= \frac{1}{3} (k+1)(k+2)(12k^2 + 32k + 21) \\
 &= \frac{1}{3} (k+1)(k+2)(2k+3)(6k+7)
 \end{aligned}$$

Hence, if true for $m=k$ then the truth of $P(k+1)$ also follows.

Step 4: $P(1)$ is true and if $P(k)$ is true then so is $P(k+1)$. So, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.

- 6 a) Find the Maclaurin series of $\ln(1 + \cos(x))$ up to, and including, the term involving x^4 . You may use without derivation that $f'''(0) = 0$ and $f^{(IV)}(0) = -\frac{1}{4}$

[5 marks]

$$f(x) = \ln(1 + \cos(x))$$

$$f'(x) = -\frac{\sin(x)}{1 + \cos(x)}$$

$$f''(x) = \frac{\sin^2(x) + \cos^2(x) + \cos(x)}{(1 + \cos(x))^2}$$

Hence,

$$f(0) = \ln(2)$$

$$f'(0) = 0$$

$$f''(0) = -\frac{1}{2}$$

$$f'''(0) = 0$$

$$f^{(IV)}(0) = -\frac{1}{4}$$

and so the Maclaurin series for $\ln(1 + \cos(x))$ is

$$\ln(2) - \frac{x^2}{4} - \frac{x^4}{96} - \dots$$

b) Hence, find $\lim_{x \rightarrow 0} \frac{\ln(1 + \cos(x)) - \ln(2)}{x^2}$

[2 marks]

Using the series in (a),

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\ln(1 + \cos(x)) - \ln(2)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\ln(2) - \frac{x^2}{4} - \frac{x^4}{96} - \ln(2)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{x^2}{4} - \frac{x^4}{96} - \dots}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{1}{4} - \frac{x^2}{96} - \dots}{1} \\ &= -\frac{1}{4} \end{aligned}$$

- c) Why could the limit in (b) have also been computed using l'Hôpital's rule?

[1 mark]

Since $\lim_{x \rightarrow 0} \ln(1 + \cos(x)) - \ln(2) = 0$ and

$\lim_{x \rightarrow 0} x^2 = 0$ the limit in (b) is an indeterminate form and so amenable to L'Hôpital's rule

7 Use the mid-ordinate rule with 4 strips to evaluate

$$\int_0^2 \frac{1}{\sqrt[3]{1+x^3}} dx, \text{ giving your answer to 4 decimal places.}$$

[4 marks]

$$h = \frac{2}{4} = \frac{1}{2}$$

$$f(x) = \frac{1}{\sqrt[3]{1+x^3}}$$

So

x	$f(x)$
0.25	0.9948452693
0.75	0.8892957031
1.25	0.6970106114
1.75	0.5397531534

Then by the mid-ordinate rule

$$A = \frac{1}{2} (0.9948452693 + 0.8892957031 + 0.6970106114 + 0.5397531534)$$

$$= 1.5604 \text{ to 4 decimal place places}$$

- 8 a) Show that the equation $9x^2 + 54x - 16y^2 + 64y = 127$ represents a hyperbola by showing that it is a translation of a canonical parabola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. State the values of a and b and the translation column vector.

[4 marks]

$$9x^2 + 54x - 16y^2 + 64y = 127$$

$$\Rightarrow 9(x^2 + 6x) - 16(y^2 - 4y) = 127$$

$$\Rightarrow 9((x+3)^2 - 9) - 16((y-2)^2 - 4) = 127$$

$$\Rightarrow 9(x+3)^2 - 16(y-2)^2 = 144$$

$$\Rightarrow \frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$$

which is a translation of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ by the

column vector $\begin{bmatrix} -3 \\ +2 \end{bmatrix}$ and $a=4, b=3$

- b) State the coordinates of the vertices of the hyperbola $9x^2 + 54x - 16y^2 + 64y = 127$ and also the equations of the asymptotes.

[4 marks]

	Original	Transformed
Vertices	(-4, 0)	(-7, 2)
	(4, 0)	(1, 2)
Asymptotes	$y = \pm \frac{3}{4}x$	$y - 2 = \pm \frac{3}{4}(x + 3)$ $\Rightarrow y = \frac{3}{4}x + \frac{19}{4}$ and $y = -\frac{3}{4}x - \frac{1}{4}$

- c) Sketch the graph of $9x^2 + 54x - 16y^2 + 64y = 127$
- [2 marks]

See Diagram

- 9 a) At time, t , hours, the number of bacteria, P , on a petri dish can be modelled by the differential equation

$$\frac{dP}{dt} = 3(40t - P),$$

Given that the initial number of bacteria is 200, show that, at time, t , the number of bacteria is given by

$$P = \frac{40}{3} (3t + 16e^{-3t} - 1).$$

[5 marks]

$$\frac{dP}{dt} = 3(40t - P)$$

$$\Rightarrow \frac{dP}{dt} = 120t - 3P$$

$$\Rightarrow \frac{dP}{dt} + 3P = 120t$$

Use integrating factor method with

$$\text{IF: } e^{\int 3 dt} = e^{3t}$$

Hence,

$$e^{3t} \frac{dP}{dt} + 3e^{3t}P = 120te^{3t}$$

$$\Rightarrow \frac{d}{dt} [e^{3t}P] = 120te^{3t}$$

Integrating both sides with respect to t (by parts or RHS)

$$e^{3t}P = 40te^{3t} - \frac{40}{3}e^{3t} + C$$

When $t = 0, P = 200$,

$$200 = 0 - \frac{40}{3} + C \Rightarrow C = \frac{640}{3}$$

and so,

$$e^{3t}P = 40te^{3t} - \frac{40}{3}e^{3t} + \frac{640}{3}$$

$$\Rightarrow P = 40t - \frac{40}{3} + \frac{640}{3e^{3t}}$$

$$= \frac{40}{3} \left(3t + 16e^{-3t} - 1 \right)$$

- b) Many differential equations can only be solved numerically. An improved Euler method for the differential equation $\frac{dP}{dt} = f(t, P)$ is given by

$$P_{r+1} = P_r + \frac{1}{2} (k_1 + k_2),$$

where $k_1 = hf(t_r, P_r)$

$$k_2 = hf(t_r + h, P_r + k_1)$$

Consider again $\frac{dP}{dt} = 3(40t - P)$ with $P(0) = 200$,

using a step size of 0.1 find an approximation to the number of bacteria present on the petri dish after 6 minutes.

[4 marks]

Let $h = 0.1$ for 6 minutes $\Rightarrow \frac{60}{10} = 6$

and $f(t, P) = 3(40t - P)$

Then

$$\begin{aligned} k_1 &= 0.1 \times 3(40 \times 0 - 200) \\ &= -60 \end{aligned}$$

and

$$k_2 = 0.1 f(0.1, 200 - 60)$$

$$\begin{aligned} &= 0.1 \times f(0.1, 140) \\ &= 0.1 \times 3(40 \times 0.1 - 140) \\ &= -40.8 \end{aligned}$$

$$P_{0.1} = 200 + \frac{1}{2}(-60 - 40 \cdot 8)$$
$$= 149.6$$

10 The cubic polynomial $x^3 + x^2 + px + q$ has roots α, β and γ .

a) Find $\alpha + \beta + \gamma$

[1 mark]

$$\alpha + \beta + \gamma = -1$$

b) Find $\alpha\beta\gamma$

[1 mark]

$$\alpha\beta\gamma = -q$$

- c) Given that $\alpha = 2 + 3i$ and that p and q are real, find the values of

i) β and γ

[2 marks]

$$\beta = 2 - 3i,$$

$$\text{Using } \alpha + \beta + \gamma = -1$$

$$2 + 3i + 2 - 3i + \gamma = -1$$

$$\Rightarrow \gamma = -5$$

ii) p and q

[3 marks]

$$\alpha\beta\gamma = q, \text{ and}$$

$$\begin{aligned} \alpha\beta\gamma &= (2 - 3i)(2 + 3i)(-5) \\ &= -65 \end{aligned}$$

$$\Rightarrow q = 65$$

And

$$\begin{aligned} p &= \alpha\beta + \alpha\gamma + \beta\gamma \\ &= (2 - 3i)(2 + 3i) + (2 + 3i)(-5) + (2 - 3i)(-5) \\ &= 13 - 10 - 15 = -10 + 15i \\ &= -7 \end{aligned}$$

- c) Find a cubic equation (with integer coefficients) which has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

[2 marks]

We have the polynomial

$$x^3 + x^2 - 7x + 65$$

Consider

\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^2 - 7\left(\frac{1}{x}\right) + 65 = 1

$$\Rightarrow 65z^3 - 7z^2 + z + 1 = 0 \quad \text{has roots } \frac{1}{\alpha}, \frac{1}{\beta}$$

and $\frac{1}{\gamma}$

11 Consider the matrix $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

- a) Given that $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of \mathbf{M} find the corresponding eigenvalue.

[3 marks]

$$\mathbf{M}v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

$$= 3v_1$$

So the corresponding eigenvalue is 3.

- b) Find the other eigenvalues, and their associated eigenvectors, for the matrix \mathbf{M} .

[6 marks]

$$\text{Consider } \mathbf{M}v = \lambda v$$

$$\Rightarrow (\mathbf{M} - \lambda I)v = 0$$

$$\Rightarrow |\mathbf{M} - \lambda I| = 0$$

$$\text{so } \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \lambda = 1, \lambda = 2, \lambda = 3$$

When $\lambda = 1$, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{array}{l} 0 = x \\ 2y = y \\ y + 2 = y + 2 \end{array}$$

$$\Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ is a suitable eigenvector}$$

When $\lambda = 2$ a suitable eigenvector is $\underline{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

c) Hence, find an expression for \mathbf{M}^n for any $n \in \mathbb{N}$.

[4 marks]

$$\mathbf{M}^n = (\mathbf{U} \mathbf{D} \mathbf{U}^{-1})^n$$

$$= \mathbf{U} \mathbf{D}^n \mathbf{U}^{-1}$$

where $\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$\mathbf{U}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

∴

$$\mathbf{M}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

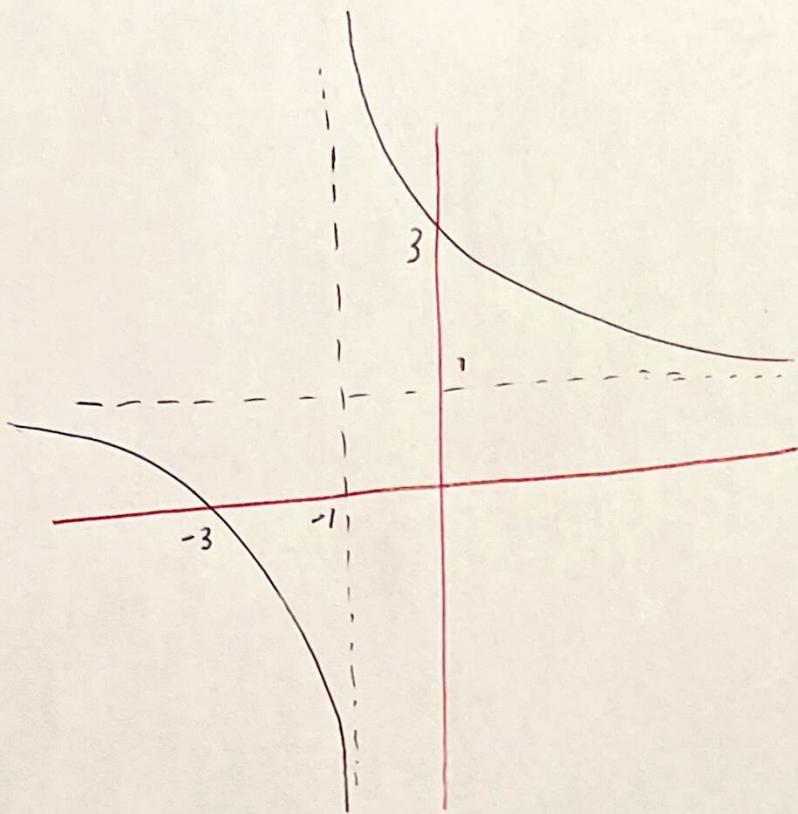
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 2^{2n} & 0 \\ 0 & 0 & 3^{3n} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & -2^{2n} & 2^n \\ 0 & 3^{3n} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^{2n} & 0 \\ 0 & 3^n - 2^n & 2^n \end{pmatrix}$$

- 12 a) Sketch the function $y = \frac{x+3}{x+1}$

[3 marks]



- b) Given that $\frac{x+3}{x+1} \leq x+3$ show that $\frac{x(x+3)}{x+1} \geq 0$.

[4 marks]

$$\frac{x+3}{x+1} \leq x+3$$

$$\Rightarrow 0 \leq x+3 - \frac{x+3}{x+1}$$

$$\Rightarrow 0 \leq \frac{(x+3)(x+1) - (x+3)}{x+1}$$

$$0 \leq \frac{x^2 + 4x + 3 - (x+3)}{x+1}$$

$$\Rightarrow 0 \leq \frac{x(x+3)}{(x+1)}$$

- c) John says that the solutions $\frac{x(x+3)}{x+1} \geq 0$ are identical to the solutions of $x(x+3)(x+1) \geq 0$. Explain why John is wrong.

[2 marks]

Because $x = -1$ is a solution of $x(x+3)(x+1) \geq 0$ but not for $\frac{x(x+3)}{(x+1)} \geq 0$ since $x = -1$ is an asymptote for $y = \frac{x(x+3)}{(x+1)}$.

d) Solve $\frac{x+3}{x+1} \leq x+3$

[2 marks]

$-3 \leq x < -1$ and $x \geq 0$

- 13 On an Indian nature reserve there are Tigers and Sambar deer. Let $x(t)$ denote the population of deer at time t , and $y(t)$ denote the population of tigers.

- The rate of increase of tigers per year equals 40 % of the number of deer.
- If there were no tigers present, then the deer population would increase by 120 % per year.
- When both tigers and deer are present then, on average, each tiger kills 1.4 deer per year.
- When $t = 0$ there are 600 deer and 100 tigers.

- a) Construct, and solve, a mathematical model for the number of deer present at time t years.

[9 marks]

$$\text{Let } x = N^{\circ} \text{ of deer}$$

$$y = N^{\circ} \text{ of tigers}$$

Then $\frac{dy}{dx} = 0.4x \quad (1)$

and $\frac{dx}{dt} = 1.4x - 1.2y \quad (2)$

$$(2) \Rightarrow 1.2y = 1.4x - \frac{dx}{dt}$$

$$\Rightarrow y = \frac{1.4}{1.2}x - \frac{1}{1.2} \frac{dx}{dt}$$

Differentiating

$$\frac{dy}{dt} = \frac{1.4}{1.2} \frac{dx}{dt} - \frac{1}{1.2} \frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{1}{1.2} \frac{d^2x}{dt^2} - \frac{1.4}{1.2} \frac{dx}{dt} + 0.4x \quad \text{using (1)}$$

$$\Rightarrow \frac{d^2x}{dt^2} - 1.4 \frac{dx}{dt} + 0.48x = 0$$

Auxiliary equation $\lambda^2 - 1.4\lambda + 0.48 = 0$
 $\Rightarrow \lambda = 0.8$ and $\lambda = 0.6$

$$\text{Hence } x = Ae^{0.8t} + Be^{0.6t} \quad (3)$$

At $t=0$ $x = 600$, so why (3)

$$600 = A + B \quad (4)$$

$$\text{Also } \frac{dx}{dt} = 0.8Ae^{0.8t} + 0.6Be^{0.6t}$$

Initial rate from (2) & $1.4 \times 600 - 1.2 \times 100 = 720$, so

$$720 = 0.8A + 0.6B \quad (5)$$

Solving (4) and (5) simultaneously,

$$A = 1800 \text{ and } B = -1200$$

Hence,

$$x = 1800e^{0.8t} - 1200e^{-0.6t}$$

- b) Explain why this model is unrealistic and suggest an improvement.

[2 marks]

The number of deer will increase. It doesn't account for external factors such as limits on food.
 Add in limiting behaviour

14 a) Find the partial fraction decomposition of $\frac{4x^2}{(x^2 + 9)^2}$

[4 marks]

$$\frac{4x^2}{(x^2 + 9)^2} = \frac{A}{x^2 + 9} + \frac{B}{(x^2 + 9)^2}$$

$$\Rightarrow 4x^2 = A(x^2 + 9) + B$$

$$\text{Let } x = 0 \Rightarrow A = 0$$

$$\text{Let } x = 3 \Rightarrow 0 = 9A + B$$

$$\Rightarrow B = -36$$

Hence,

$$\frac{4x^2}{(x^2 + 9)^2} = \frac{0}{x^2 + 9} - \frac{36}{(x^2 + 9)^2}$$

- b) A gradually tapered vase is milled out of aluminium so that its profile is the rotation of the curve $y = \frac{2x}{x^2 + 9}$ about the x -axis between the lines $x = \sqrt{3}$ and $x = 3\sqrt{3}$.

Show that the volume of the vase is $\frac{\pi^2}{9}$ cubic units.

[10 marks]

You can use, without proof, that

$$\frac{1}{3} \sin \left(2 \arctan \left(\frac{x}{3} \right) \right) = \frac{2x}{x^2 + 9}$$

$$\text{Volume} = \pi \int_{\sqrt{3}}^{3\sqrt{3}} y^2 dx$$

$$= \pi \int_{\sqrt{3}}^{3\sqrt{3}} \left(\frac{2x}{x^2 + 9} \right)^2 dx$$

$$= \pi \int_{\sqrt{3}}^{3\sqrt{3}} \frac{4x^2}{(x^2 + 9)^2} dx$$

$$= \pi \int_{\sqrt{3}}^{3\sqrt{3}} \frac{4}{x^2 + 9} - \frac{36}{(x^2 + 9)^2} dx$$

Now $\int \frac{4}{x^2 + 9} dx = \frac{4}{3} \arctan \left(\frac{x}{3} \right) + C$

and

$$\int \frac{36}{(x^2+9)^2} dx = 36 \int \frac{1}{(x^2+9)^2} dx$$

Let $x = 3\tan(s)$ then $\frac{dx}{ds} = 3\sec^2(s) \Rightarrow ds = \frac{1}{3\sec^2(s)} dx$

$$\text{and } (x^2+9)^2 = (9\tan^2(s)+9)^2 = 81\sec^4(s).$$

Also $s = \arctan\left(\frac{x}{3}\right)$, so

$$36 \int \frac{1}{(x^2+9)^2} dx = 36 \int \frac{1}{81\sec^4(s)} 3\sec^2(s) ds$$

$$= \frac{108}{81} \int \frac{1}{\sec^2(s)} ds$$

$$= \frac{108}{81} \int \cos^2(s) ds$$

$$= \frac{4}{3} \int \cos^2(s) ds$$

$$= \frac{2}{3} \int \cos(2s) + 1 ds$$

$$= \frac{1}{3} \sin(2s) + \frac{2}{3}s + C$$

$$= \frac{1}{3} \sin\left(2\arctan\left(\frac{x}{3}\right)\right) + \frac{2}{3} \arctan\left(\frac{x}{3}\right)$$

$$= \frac{2x}{x^2+9} + \frac{2}{3} \arctan\left(\frac{x}{3}\right)$$

Hence, the volume is given by

$$\pi \int_{\sqrt{3}}^{3\sqrt{3}} \left(\frac{2x}{x^2+9} \right)^2 dx$$

$$= \pi \left[\frac{4}{3} \arctan\left(\frac{x}{3}\right) - \frac{2x}{x^2+9} - \frac{2}{3} \arctan\left(\frac{x}{3}\right) \right]_{\sqrt{3}}^{3\sqrt{3}}$$

$$= \pi \left[\frac{2}{3} \arctan\left(\frac{x}{3}\right) - \frac{2x}{x^2+9} \right]_{\sqrt{3}}^{3\sqrt{3}}$$

$$= \frac{\pi^2}{9}$$