AQA A-Level Further Maths 2022 Paper 2

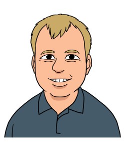
Do nut turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100



1 Let $z = 3 - 2\mathrm{i}$. Find $\dfrac{z^2}{z^*}$

$3 + 2\mathrm{i}$ $\quad -\dfrac{1}{13}(9 + 46\mathrm{i})$ $\quad \dfrac{1}{13}(-9 + 46\mathrm{i})$ $\quad \dfrac{1}{13}(5 + 12\mathrm{i})$

[1 mark]

2 Find the matrix representing an anticlockwise rotation by $30°$ centre the origin, followed by a reflection in the line $y = x$.

$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ $\quad \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ $\quad \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ $\quad \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

[1 mark]

3 Let M be the value of the mean value of the function $f(x)$ over the interval $[a, b]$. Then the mean value of the function $f(x) + d$ over the interval $[a, b]$ is equal to:

M $\quad$ $M + (b - a)$ $\quad$ $M + d$ $\quad$ $M + d(b - a)$

[1 mark]

4 Sketch the function $y = \text{sech}(x)$

[1 mark]

5 Prove by induction that the following statement is true for all $n \in \mathbb{N}$.

$$\sum_{r=1}^{2n} r^2(r+1) = \frac{1}{3}n(n+1)(2n+1)(6n+1)$$

[7 marks]

6 **a)** Find the Maclaurin series of $\ln\left(1+\cos(x)\right)$ up to, and including, the term involving x^4. You may use without derivation that $f'''(0)=0$ and $f^{(IV)}(0)=-\dfrac{1}{4}$

[5 marks]

b) Hence, find $\lim_{x \to 0} \dfrac{\ln\left(1 + \cos(x)\right) - \ln(2)}{x^2}$

[2 marks]

c) Why could the limit in **(b)** have also been computed using l'Hôpital's rule?

[1 mark]

7 Use the mid-ordinate rule with 4 strips to evaluate $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{1+x^3}}\,\mathrm{d}x$, giving your answer to 4 decimal places.

[4 marks]

8 **a)** Show that the equation $9x^2 + 54x - 16y^2 + 64y = 127$ represents a hyperbola by showing that it is a translation of a canonical parabola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. State the values of a and b and the translation column vector.

[4 marks]

b) State the coordinates of the vertices of the hyperbola $9x^2 + 54x - 16y^2 + 64y = 127$ and also the equations of the asymptotes.

[4 marks]

c) Sketch the graph of $9x^2 + 54x - 16y^2 + 64y = 127$

[2 marks]

9 **a)** At time, t, hours, the number of bacteria, P, on a petri dish can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 3\,(40t - P),$$

Given that the initial number of bacteria is 200, show that, at time, t, the number of bacteria is given by
$P = \dfrac{40}{3}\left(3t + 16\mathrm{e}^{-3t} - 1\right).$

[5 marks]

b) Many differential equations can only be solved numerically. An improved Euler method for the differential equation $\dfrac{\mathrm{d}P}{\mathrm{d}t} = f(t, P)$ is given by

$$P_{r+1} = P_r + \frac{1}{2}\left(k_1 + k_2\right),$$

where
$$k_1 = hf\left(t_r, P_r\right)$$
$$k_2 = hf\left(t_r + h, P_r + k_1\right)$$

Consider again $\dfrac{\mathrm{d}P}{\mathrm{d}t} = 3\,(40t - P)$ with $P(0) = 200$, using a step size of 0.1 find an approximation to the number of bacteria present on the petri dish after 6 minutes.

[4 marks]

10 The cubic polynomial $x^3 + x^2 + px + q$ has roots α, β and γ.

a) Find $\alpha + \beta + \gamma$

[1 mark]

b) Find $\alpha\beta\gamma$

[1 mark]

c) Given that $\alpha = 2 + 3\mathrm{i}$ and that p and q are real, find the values of

i) β and γ

[2 marks]

ii) p and q

[3 marks]

c) Find a cubic equation (with integer coefficients) which has roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$.

[2 marks]

11 Consider the matrix $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

a) Given that $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$ find the corresponding eigenvalue.

[3 marks]

b) Find the other eigenvalues, and their associated eigenvectors, for the matrix $\mathbf{M}$.

[6 marks]

c) Hence, find an expression for $\mathbf{M}^n$ for any $n \in \mathbb{N}$.

[4 marks]

12 a) Sketch the function $y = \dfrac{x+3}{x+1}$

[3 marks]

b) Given that $\dfrac{x+3}{x+1} \leq x+3$ show that $\dfrac{x(x+3)}{x+1} \geq 0$.

[4 marks]

c) John says that the solutions $\dfrac{x(x+3)}{x+1} \geq 0$ are identical to the solutions of $x(x+3)(x+1) \geq 0$.
Explain why John is wrong.

[2 marks]

d) Solve $\dfrac{x+3}{x+1} \leq x+3$

[2 marks]

13 On an Indian nature reserve there are Tigers and Sambar deer. Let $x(t)$ denote the population of deer at time t,and $y(t)$ denote the population of tigers.

- The rate of increase of tigers per year equals $40\,\%$ of the number of deer.
- If there were no tigers present, then the deer population would increase by $120\,\%$ per year.
- When both tigers and deer are present then,on average, each tiger kills 1.4 deer per year.
- When $t = 0$ there are 600 deer and 100 tigers.

a) Construct, and solve, a mathematical model for the number of deer present at time t years.

[9 marks]

b) Explain why this model is unrealistic and suggest an improvement.

[2 marks]

14 a) Find the partial fraction decomposition of $\dfrac{4x^2}{\left(x^2+9\right)^2}$

[4 marks]

b) A gradually tapered vase is milled out of aluminium so t hat it's profile is the rotation of the curve $y = \dfrac{2x}{x^2+9}$ about the $x-$axis between the lines $x = \sqrt{3}$ and $x = 3\sqrt{3}$.

Show that the volume of the vase is $\dfrac{\pi^2}{9}$ cubic units.

[10 marks]

You can use, without proof, that

$$\frac{1}{3}\sin\left(\arctan\left(\frac{x}{3}\right)\right) = \frac{2x}{x^2+9}$$