

Solutions

AQA A-Level Further Maths 2022 Paper 1

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100



- 1 Let $z = 3 \left(\cos \left(\frac{\pi}{4} \right) + i \sin \left(\frac{\pi}{4} \right) \right)$ and
 $w = 4 \left(\cos \left(\frac{7\pi}{6} \right) + i \sin \left(\frac{7\pi}{6} \right) \right)$ then zw equals

$$12 \left(\cos \left(\frac{17\pi}{12} \right) + i \sin \left(\frac{17\pi}{12} \right) \right)$$

$$7 \left(\cos \left(\frac{17\pi}{12} \right) + i \sin \left(\frac{17\pi}{12} \right) \right)$$

$$12 \left(\cos \left(\frac{-11\pi}{12} \right) + i \sin \left(\frac{-11\pi}{12} \right) \right)$$

$$7 \left(\cos \left(\frac{-11\pi}{12} \right) + i \sin \left(\frac{-11\pi}{12} \right) \right)$$

[1 mark]

- 2 The derivative, with respect to x of $\cosh(4x)$ is

$\sinh(4x)$

$4 \sinh(4x)$

$4 \cosh(4x)$

$2 \sinh(4x)$

[1 mark]

- 3 A particle performing simple harmonic motion has a speed of 5 ms⁻¹ when it is 2 m away from the centre of oscillation. If the amplitude is 4 m, what is the period of oscillation?

$$\frac{12\pi}{5\sqrt{3}}$$

$$\frac{10\sqrt{3}}{6}$$

$$\frac{12}{5\sqrt{3}}$$

$$\frac{10\pi}{\sqrt{3}}$$

[1 mark]

- 4 a) Prove that the logarithmic form of $y = \text{arsinh}(x)$ is

$$y = \ln \left(x + \sqrt{x^2 + 1} \right)$$

[4 marks]

$$y = \text{arsinh}(x) \Rightarrow \sinh(y) = x$$

$$\Rightarrow \frac{e^y - e^{-y}}{2} = x$$

$$\Rightarrow 2x = e^y - e^{-y}$$

$$\Rightarrow 2xe^y = e^{2y} - 1$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

by the quadratic formula

$$\Rightarrow e^y = x + \sqrt{x^2 + 1}$$

where we discard the - sign
 $\sqrt{x^2 + 1} \geq x$ and $x \geq 0 \Rightarrow y$.
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$$\text{so } y = \ln(x + \sqrt{x^2 + 1})$$

$$\text{Hence, } \operatorname{arshinh}(x) = \ln(x + \sqrt{x^2 + 1})$$

- b) Solve $3 \cosh^2(x) - 7 \sinh(x) - 9 = 0$, giving your answers in exact logarithmic form.

[4 mark]

$$3 \cosh^2(x) - 7 \sinh(x) - 9 = 0$$

$$\Rightarrow 3(1 + \sinh^2(x)) - 7 \sinh(x) - 9 = 0$$

$$\Rightarrow 3 \sinh^2(x) - 7 \sinh(x) - 6 = 0$$

$$\Rightarrow (3 \sinh(x) + 2)(\sinh(x) - 3) = 0$$

$$\Rightarrow \sinh(x) = -\frac{2}{3} \text{ or } \sinh(x) = 3$$

and so,

$$x = \ln(3 + \sqrt{10}) \text{ or } x = \ln\left(-\frac{2}{3} + \sqrt{\frac{13}{9}}\right)$$

- 5 a) Show that $z = 2 + i$ is a root of the polynomial

$$p(z) = z^4 - 4z^3 + 14z^2 - 36z + 45.$$

[2 marks]

$$\begin{aligned} p(2+i) &= (2+i)^4 - 4(2+i)^3 + 14(2+i)^2 - 36(2+i) + 45 \\ &= (-7+24i) - 4(2+1i) + 14(3+4i) - 36(2+i) + 45 \\ &= 0 \end{aligned}$$

Hence, $2+i$ is a root of the polynomial

- b) State another complex root of $p(z)$.

[1 mark]

$$2-i$$

- c) Hence, show that the two remaining roots of $p(z)$ are purely imaginary.

[4 marks]

$$(2-2-i)(\cancel{2}-2+i) = 2^2 - 4i2 + 5$$

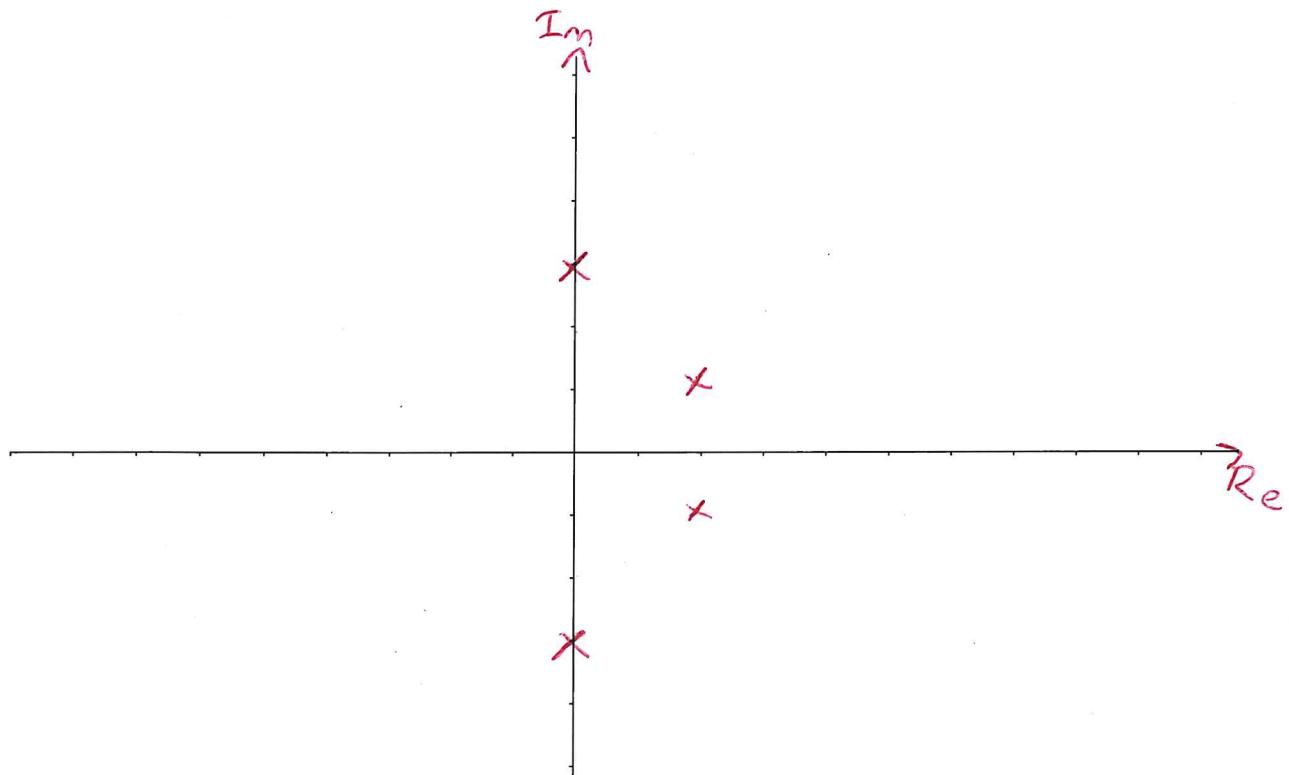
Then

$$z^4 - 4z^3 + 14z^2 - 36z + 45 = (z^2 - 4z + 5)(z^2 + 9)$$

So, the other roots are $z = \pm 3i$

- d) Plot the four roots on the Argand diagram below.

[2 marks]



- 6 Consider the matrix $T = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$

- a) Find $|T|$ and explain its geometrical significance

[2 marks]

$$\begin{aligned}|T| &= 5 \times 4 - 3 \times 2 \\ &= 14\end{aligned}$$

The area scale factor of the transformation represented by T is 16.

- b) Find the eigenvalues, and associated eigenvectors of the 2×2 matrix T .

[5 marks]

Consider $\det(T - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(4-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda + 20 - 6 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda + 14 = 0$$

$$\Rightarrow (\lambda-7)(\lambda-2) = 0$$

So the eigenvalues are $\lambda=7$ and $\lambda=2$

When $\lambda=7$: $T\underline{v} = \lambda\underline{v}$

$$(5 \ 3) \begin{pmatrix} x \\ y \end{pmatrix} = 7 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 5x + 3y = 7x \\ 2x + 4y = 7y \end{cases} \Rightarrow \begin{cases} 3y = 2x \\ y = \frac{2}{3}x \end{cases}$$

let $x=1$, then an eigenvector

$$\therefore \underline{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

When $\lambda=2$:

$$T\underline{v} = \lambda\underline{v}$$

$$(5 \ 3) \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 5x + 3y = 2x \\ 2x + 4y = 2y \end{cases} \Rightarrow \begin{cases} 3y = -3x \\ 2x = -2y \end{cases} \Rightarrow y = -x$$

$$\therefore \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

- c) Describe the relationship between the eigenvectors of a matrix and the invariant lines of the same matrix.

The eigenvectors are in the direction of the invariant lines [1 mark]

- d) Find, with full justification, the invariant lines of $T = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$ [4 marks]

$$\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Rightarrow \begin{cases} x' = 5x + 3(mx + c) \\ y' = 2x + 4(mx + c) \end{cases}$$

$$\text{but } y' = mx'' + c$$

$$\Rightarrow 2x + 4mx + 4c = m(5x + 3mx + 3c) + c$$

$$\Rightarrow 2x + 4mx + 4c = 5mx + 3m^2x + 3mc + c$$

$$\Rightarrow 3m^2x + mx - 2x + 3mc + c - 4c = 0$$

$$\Rightarrow (3m^2 + m - 2)x + c(3m - 3) = 0$$

$$\Rightarrow (3m - 2)(m + 1)x + c(3m - 3) = 0$$

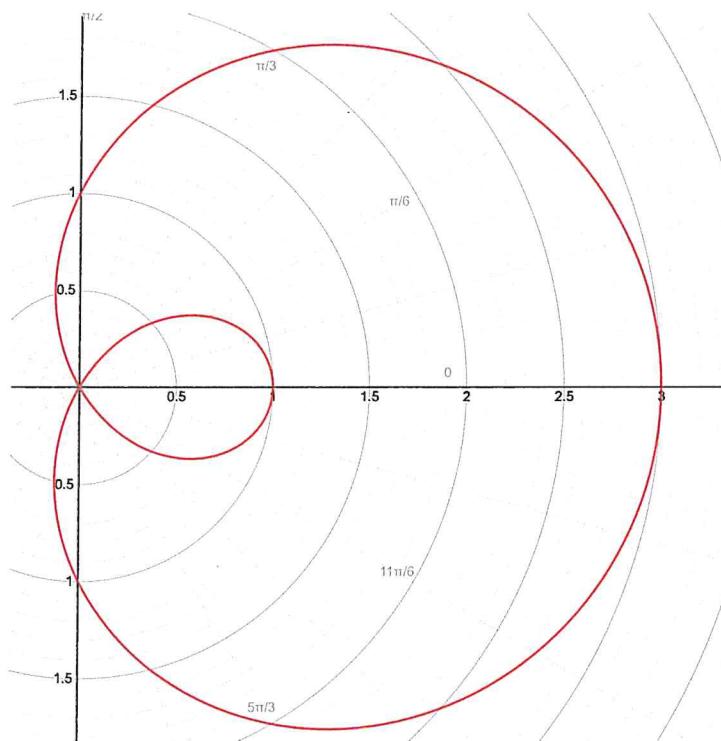
$$\text{Hence } m = \frac{2}{3} \text{ or } m = -1$$

When $m = \frac{2}{3}$, $c = 0$

When $m = -1$, $c = 0$

So the invariant lines are $y = \frac{2}{3}x$ or $y = -x$

- 7 a) Sketch the curve $r = 1 + 2 \cos(\theta)$



[2 marks]

- b) Find the total area contained between the outer and inner loops of the curve sketched in (a)

[7 marks]

$$\begin{aligned}
 \text{Inner Loop } A_1 &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2} r^2 d\theta \\
 &= \frac{1}{2} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1+2\cos(\theta))^2 d\theta \\
 &= \frac{1}{2} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1+4\cos(\theta)+4\cos^2(\theta)) d\theta \\
 &= \frac{1}{2} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (3+4\cos(\theta)+2\cos(2\theta)) d\theta
 \end{aligned}$$

but $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\sin(2\theta) \right]_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

Similarly,

$$\text{Outer Loop: } A_2 = 2 \times \frac{1}{2} \int_0^{\frac{2\pi}{3}} r^2 d\theta$$

$$= \int_0^{\frac{2\pi}{3}} (1+2\cos(\theta))^2 d\theta$$

$$= \left[3\theta + 4\sin(\theta) + \sin(2\theta) \right]_0^{\frac{2\pi}{3}}$$

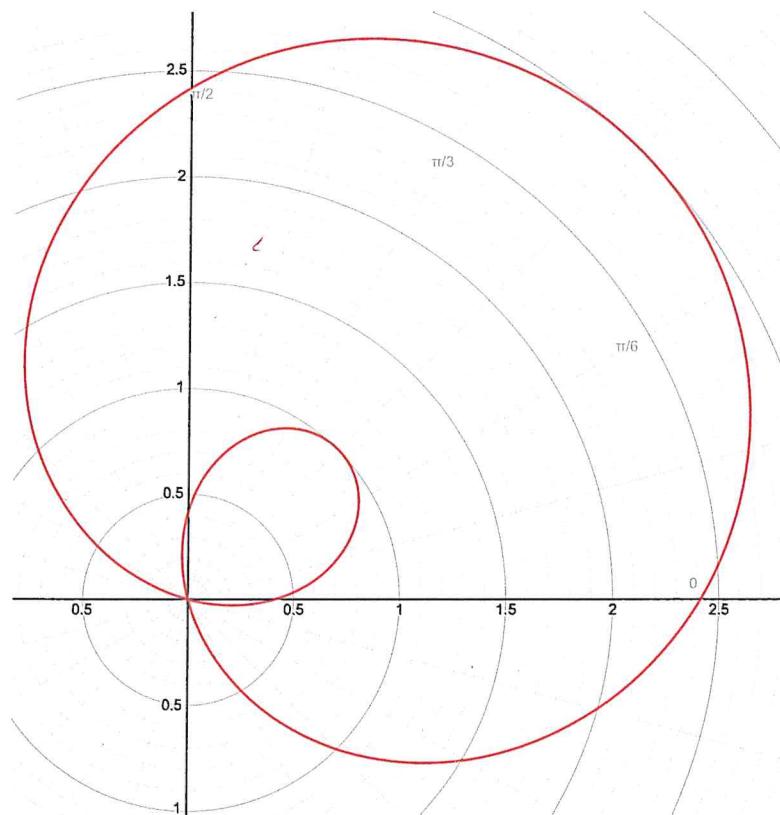
$$= 2\pi + \frac{3\sqrt{3}}{2}$$

Hence, the area between the two loops is given by

$$\frac{2\pi + 3\sqrt{3}}{3} - \left(\pi - \frac{3\sqrt{3}}{2} \right) = \pi + 3\sqrt{3}$$

- c) Sketch the curve $r = 1 + 2\cos\left(\theta - \frac{\pi}{4}\right)$

[2 marks]



- 8 a) Find the angle between the line, l , with equation

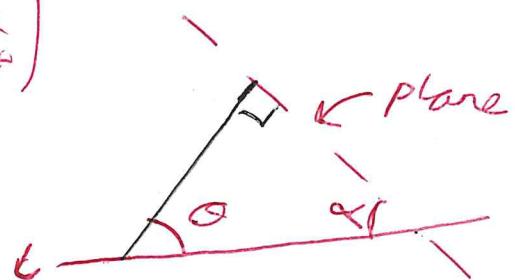
$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

$$\mathbf{r} \cdot (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 4$$

[5 marks]

Normal to the plane is $\mathbf{d} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Direction of line is $\mathbf{d} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$



Then

$$\mathbf{d} \cdot \mathbf{d} = |\mathbf{d}| |\mathbf{d}| \cos(\theta)$$

$$\Rightarrow \cos(\theta) = \frac{\left(\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right)}{\sqrt{\left(\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \right)^2} \sqrt{\left(\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right)^2}}$$

$$= \frac{8}{\sqrt{14} \sqrt{29}}$$

$$\therefore \theta = \arccos\left(\frac{8}{\sqrt{14} \sqrt{29}}\right)$$

$$= 66.61^\circ$$

Hence $\alpha = 90^\circ - 66.61^\circ = 23.39^\circ$ is the angle between the plane and the line

- b) Find the shortest distance from the point $P(2,2,4)$ to the plane with equation $2x + 3y - z - 4 = 0$

[4 marks]

The line from P to a point M on the plane has equation

$$\text{L} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

Substitute this into the equation of the plane

$$2(2+2\lambda) + 3(2+3\lambda) - (4-\lambda) - 4 = 0$$

$$\Rightarrow 4 + 4\lambda + 6 + 9\lambda - 4 + \lambda - 4 = 0$$

$$\Rightarrow 14\lambda + 2 = 0$$

$$\Rightarrow \lambda = \cancel{-\frac{1}{7}} \quad \frac{1}{7}$$

Hence, point of intersection is

$$M\left(\frac{5}{3}, \frac{3}{2}, \frac{25}{6}\right) \quad M\left(\frac{12}{7}, \frac{11}{7}, \frac{29}{7}\right)$$

Then,

$$|\vec{PM}| = \sqrt{(2 - \frac{12}{7})^2 + (2 - \frac{11}{7})^2 + (4 - \frac{29}{7})^2}$$

$$= \cancel{\sqrt{14}} \quad \cancel{\sqrt{14}} \quad \frac{2}{7}$$

$$= \cancel{\sqrt{14}} \quad \frac{\sqrt{14}}{7}$$

- 9 Find the possible angles between consecutive roots of the equation

$$1 + z + z^2 + z^3 + z^4 + z^5 = 0$$

[10 marks]

Consider the LHS

$$(1 + z + z^2 + z^3 + z^4 + z^5) = \frac{z^6 - 1}{z - 1}$$

$$\text{Hence } z^6 - 1 = 0$$

$$\Rightarrow z^6 = 1$$

$$\Rightarrow z^6 = e^{\frac{2k\pi i}{6}}$$

$$\Rightarrow z = e^{\frac{2k\pi i}{6}}$$

$$= e^{\frac{\pi k i}{3}}, k = 1, 2, 3, 4, 5$$

We discount the $k=0$ case as this is a spurious root since it's not a solution of $\frac{z^6 - 1}{z - 1}$ due to the division by $z - 1$.

$$\begin{aligned} \underline{k=1} : z &= e^{\frac{\pi i}{3}} \\ &= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) \\ &= \frac{1}{2} + i\frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \underline{k=2} : z &= e^{\frac{2\pi i}{3}} \\ &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \end{aligned}$$

$$\underline{k=3} \quad z = e^{\frac{3\pi i}{3}}$$

$$= \cos(\pi) + i \sin(\pi)$$

$$= -1$$

$$\underline{k=4}: \quad z = e^{\frac{4\pi i}{6}} = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}$$

$$\underline{k=5}: \quad z = e^{\frac{5\pi i}{6}} = \cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)$$

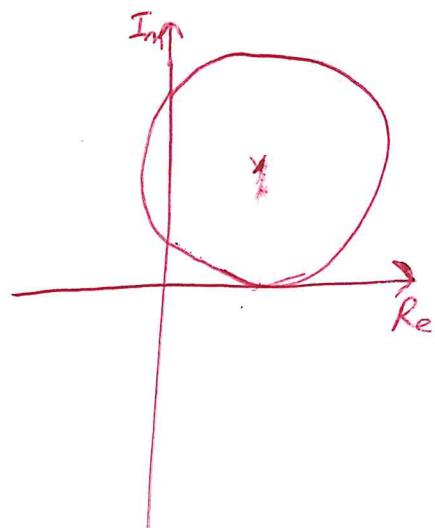
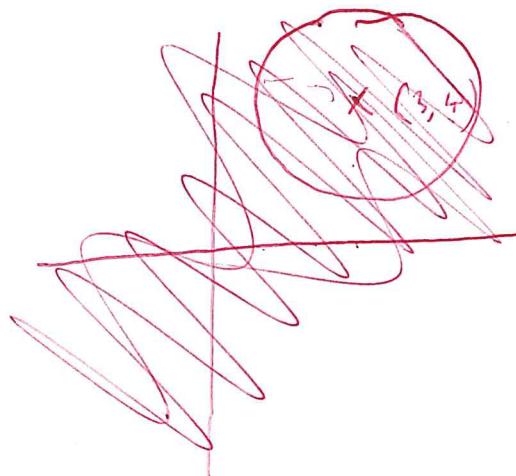
so the possible angles between the roots are

$$\frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

- 10 A complex number, z , is represented by the point P in the complex plane.

a) Given that $|z - 3 - 4i| = 4$, sketch the locus of P .

[2 marks]



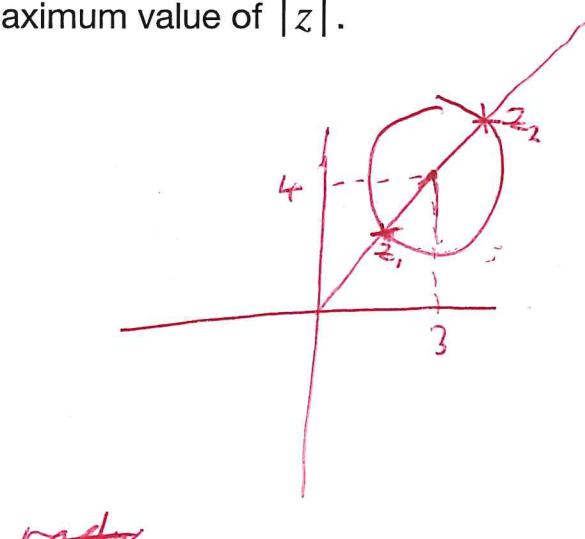
b) Write down the Cartesian equation of this locus.

[1 mark]

$$(x - 3)^2 + (y - 4)^2 = 16$$

c) Find, justifying your reasoning, the minimum value of $|z|$ and the maximum value of $|z|$.

[4 marks]



Distance from the origin to the centre is 5.

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Radius of circle is ~~5~~ 4

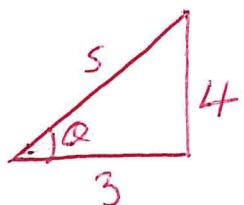
z_1 is location of the minimum value of $|z|$ and

$$|z|_{\min} = \cancel{5} 1$$

z_2 is the location of the maximum value of $|z|$ and $|z|_{\max} = \cancel{5} 9$

- d) Write down, in Cartesian form, the coordinates of the points where these minimum and maximum values of $|z|$ occur.

These points lie on the line joining the origin to the centre of the circle.



$$\sin(\theta) = \frac{4}{5}$$

$$\cos(\theta) = \frac{3}{5}$$

Hence we have

$$z_1 = \cancel{5} \left(\cos(\theta) + i \sin(\theta) \right)$$

$$= \cancel{5} \left(\frac{3}{5} + i \frac{4}{5} \right)$$

$$= \cancel{\frac{5}{5}} \cancel{5} \left(\frac{3}{5} + i \frac{4}{5} \right)$$

and $z_2 = \cancel{9} \left(\frac{3}{5} + i \frac{4}{5} \right)$

$$= \cancel{\frac{9}{5}} \cancel{5} + \cancel{\frac{36}{5}} i$$

- 11 a) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & a \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$|A| = 5(a-3)$$

[5 marks]

Inverse $A^{-1} = \frac{1}{5(a-3)} \begin{pmatrix} -5 & 4-a & 3a-2 \\ -5 & a-2 & 2a+1 \\ 5 & 1 & -7 \end{pmatrix}$

- b) Consider the following three planes:

$$\Pi_1: x + 2y + 3z = 2$$

$$\Pi_2: -2x + 3y + z = 5$$

$$\Pi_3: x + y + 2z = 4$$

- i) Explain why, using part (a), that these planes do not intersect at a unique point.

~~When~~ The intersection can be represented as the solution of

$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

but by part (a) when $\alpha = 3$, the matrix is singular and has no inverse. Hence the planes don't intersect at a unique point

- ii) What is the geometric configuration of these three planes?

[4 marks]

$$x + 2y + 3z = 2 \quad ①$$

$$-2x + 3y + z = 5 \quad ②$$

$$x + y + 2z = 4 \quad ③$$

$$\begin{aligned} ① - 3 \times ② & \quad 7x - 7y = -13 & \Rightarrow 7(x-y) = -13 \\ & & \Rightarrow x-y = -\frac{13}{7} \end{aligned}$$

$$\begin{aligned} ③ - 2 \times ② & \quad 5x - 5y = -6 & \Rightarrow 5(x-y) = -6 \\ & & \Rightarrow x-y = -\frac{6}{5} \end{aligned}$$

So the equations aren't consistent and hence the planes form a triangular prism.

- 12 A particle, P , of mass m , is suspended from a fixed point O by a light elastic string of natural length 2 m and modulus of elasticity $40m$.*

When the particle hangs in equilibrium vertically below O , the extension of the string is $\frac{g}{20}$ m.

At time $t = 0$ s, the particle is projected vertically downwards from the equilibrium position with speed $2U$.

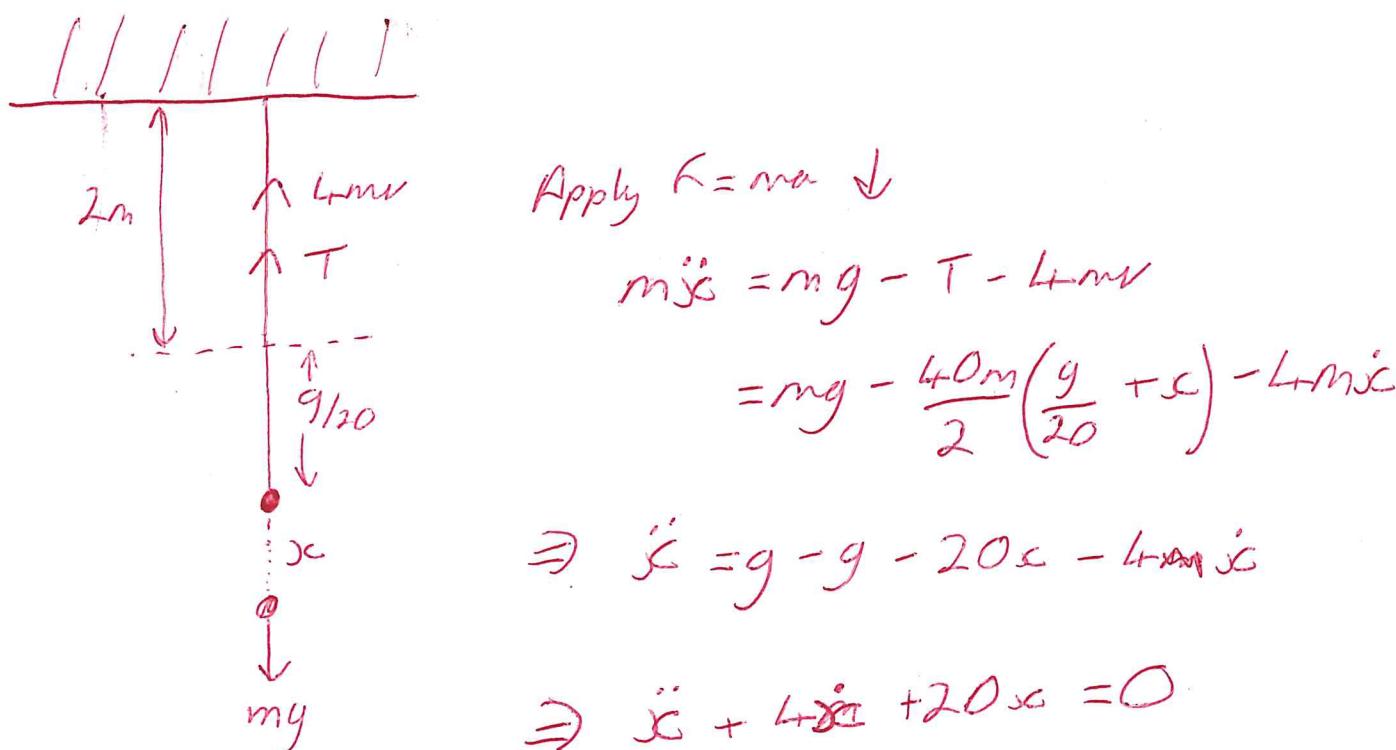
During its subsequent motion, when P is moving with speed v it experiences a resistive force of $4mv$.

The displacement of P below its equilibrium position at time t is x .

- a) Show that while the string is taut x satisfies the equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 0$$

[5 marks]



b) Find x in terms of U and t .

[7 marks]

Auxiliary Equation:

$$m^2 + 4m + 20 = 0$$

$$\Rightarrow m = -2 - 4i \quad \text{and} \quad -2 + 4i$$

Hence,

$$x(t) = e^{-2t} (A \cos(4t) + B \sin(4t))$$

When $t=0$, $x(t)=0$, hence

$$0 = e^{-2t}(A)$$

$$\Rightarrow A=0$$

$$\text{so } x = e^{-2t} B \sin(4t)$$

Then

$$\begin{aligned} x' &= B e^{-2t} [4 \cos(4t) + 4 \sin(4t)] e^{-2t} \\ &= 4B e^{-2t} \cos(4t) - 2B e^{-2t} \sin(4t) \end{aligned}$$

When $t=0$, $x=2U$

$$2U = 4B e^0 \cos(0) - 2B e^0 \sin(0)$$

$$\Rightarrow 2U = 4B$$

$$\Rightarrow B = \frac{U}{2}$$

Hence, $x = \frac{U}{2} e^{-2t} \sin(4t)$

- c) What kind of damping is being caused by this resistive force?

[1 mark]

Light damping

- d) Find the speed, in terms of U and π , of P when it first returns to its equilibrium position.

[4 marks]

When it returns to its equilibrium position, $x = 0$,

so

$$\theta = \frac{U}{2} e^{-2t} \sin(4t)$$

$$\Rightarrow \sin(4t) = 0$$

$$\Rightarrow 4t = 0, \pi$$

$$\therefore t = \frac{\pi}{4}$$

$$\dot{x} = \frac{U}{2} e^{-2t} \cos(4t) - \frac{2U}{2} e^{-2t} \sin(4t)$$

$$\text{when } t = \frac{\pi}{4}$$

$$\dot{x} = 2U e^{-\frac{\pi}{2}} \cos(\pi) - U e^{-\frac{\pi}{2}} \sin(\pi)$$

$$= -2U e^{-\frac{\pi}{2}}$$

But as we want the speed, the speed is $2U e^{-\frac{\pi}{2}}$