

AQA A-Level Maths 2022 Paper 1

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 100

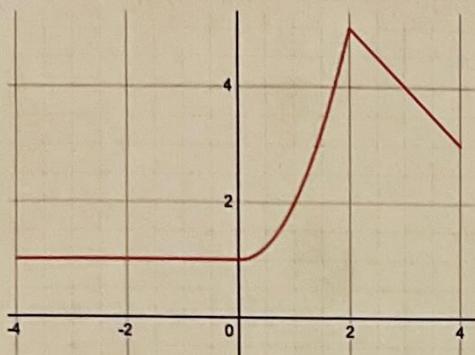
Solutions

- 1 If $(2x + 3)$ is a factor of $p(x) = ax^3 + bx^2 + cx + d$, then

$$p\left(\frac{3}{2}\right) = 0 \quad p\left(\frac{2}{3}\right) = 0 \quad \textcircled{p\left(-\frac{3}{2}\right) = 0} \quad p\left(-\frac{2}{3}\right) = 0$$

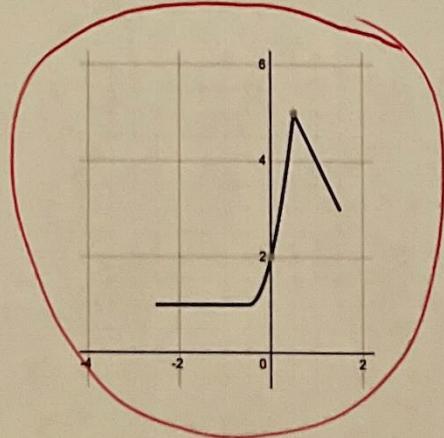
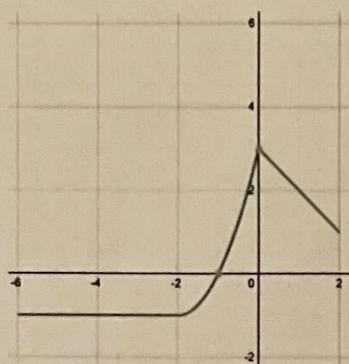
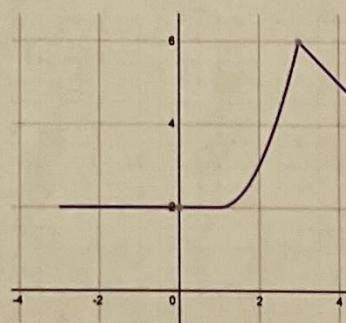
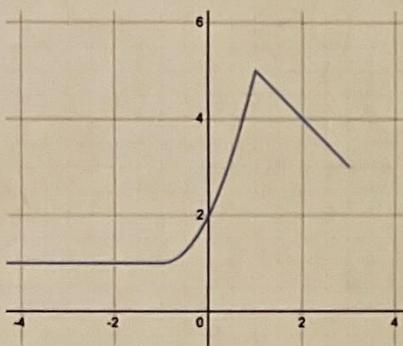
[1 mark]

- 2 The function $f(x)$ is shown below



Which of the following graphs could not be a transformation of the form $f(x + a) + b$?

(1 mark)



- 3 a) Find the Cartesian equation of the curve with parametric equations $x = -3 + 2\cos(\theta)$, $y = 4 + 2\sin(\theta)$, $0 \leq \theta \leq 2\pi$. [4 marks]

$$\cos(\theta) = \frac{x+3}{2} \quad ① \quad \sin(\theta) = \frac{y-4}{2} \quad ②$$

Since $\cos^2(\theta) + \sin^2(\theta) = 1$, using ① and ②

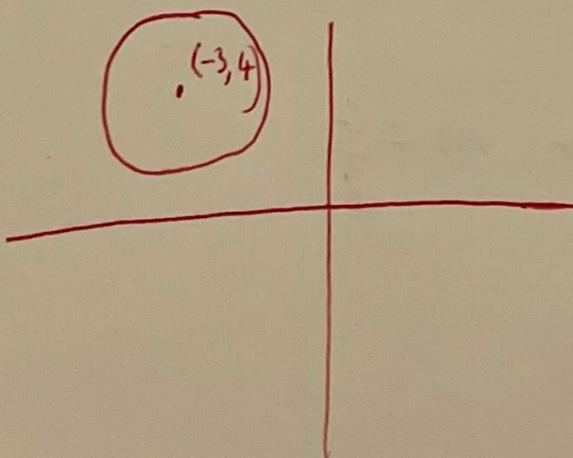
$$\left(\frac{x+3}{2}\right)^2 + \left(\frac{y-4}{2}\right)^2 = 1$$

$$\Rightarrow (x+3)^2 + (y-4)^2 = 4$$

(circle centre $(-3, 4)$, radius 2)

- b) Sketch this curve.

[1 mark]



- 4 a) The line $y = mx + 15$ is tangent to the circle $(x - 2)^2 + (y - 1)^2 = 100$. Find the possible values of m . [5 marks]

$y = mx + 15$ is a tangent if it intersects the circle at 1 point.

Let

$$y = mx + 15 \quad (1)$$

$$(x - 2)^2 + (y - 1)^2 = 100 \quad (2)$$

Substitute (1) into (2)

$$(x - 2)^2 + (mx + 14)^2 = 100$$

$$\Rightarrow x^2 - 4x + 4 + m^2x^2 + 28mx + 196 = 100$$

$$\Rightarrow (m^2 + 1)x^2 + (28m - 4)x + 100 = 0$$

As a tangent the discriminant = 0.

Now,

$$b^2 - 4ac = 0, \text{ where } a = m^2 + 1 \\ b = 28m - 4 \\ c = 100$$

$$\text{So, } (28m - 4)^2 - 4 \times (m^2 + 1) \times 100 = 0$$

$$\Rightarrow 784m^2 - 224m + 16 - 400m^2 - 400 = 0$$

$$\Rightarrow 384m^2 - 224m - 384 = 0$$

$$\Rightarrow m = \frac{-3}{24} \text{ or } m = \frac{4}{3}$$

- b) Find the coordinates of the point of intersection of the possible tangents.

[4 marks]

- 5 The fifth term of an arithmetic sequence is 22, and the tenth term of the same sequence is 37.

Find the maximum number of terms, n , such that the partial sum $S_n < 200$.

[7 marks]

$$22 = a + 4d \quad (1)$$

$$37 = a + 9d \quad (2)$$

Solving (1) and (2) simultaneously gives

$$a = 10, d = 3$$

So $S_n = \frac{1}{2}n[2a + (n-1)d]$

$$\Rightarrow 200 = \frac{1}{2}(20 + (n-1)3)$$

$$\Rightarrow \frac{1}{2}(17 + 3n) = 200$$

$$\Rightarrow \frac{3n^2}{2} + \frac{17n}{2} - 200 = 0$$

So $n = 9.05$ or -14.7

Discount the negative solution, so the maximum number of terms, n , for the sum to be less than 200 is $n = 9$

- 6 The sequence generated by the recurrence relation $u_{k+1} = pu_k + q$,
 $u_1 = 3$, where p is constant and $q = 6$ is periodic with order 2. Find
the value of p . [4 marks]

$$\begin{aligned} u_1 &= 3 \\ u_2 &= p(3) + 6 = 3p + 6 \\ u_3 &= p(3p + 6) + 6 = 3 \end{aligned}$$

More,

$$\begin{aligned} 3p^2 + 6p + 3 &= 0 \\ \Rightarrow p &= -1 \quad [3(p^2 + 2p + 1) = 0 \\ &\quad 3(p + 1)^2 = 0] \end{aligned}$$

- 7 Fully justifying your answer, solve the equation,
 $\cos^2(x) + \cos^3(x) + \cos^4(x) + \dots = 1 + \cos(x)$ for $0 < x < 180$.
[10 marks]

The LHS is a geometric sequence with first term $\cos^2(x)$ and common ratio $r = \cos(x)$. Since 0° and 180° aren't included in the range of solutions we know that $|r| = |\cos(x)| < 1$, and so we have a convergent infinite series with sum

$$\begin{aligned} S_{\infty} &= \frac{a}{1-r} \\ &= \frac{\cos^2(x)}{1 - \cos(x)} \end{aligned}$$

Hence,

$$\frac{\cos^2(x)}{1 - \cos(x)} = 1 + \cos(x)$$

$$\Rightarrow \cos^2(x) = (1 + \cos(x))(1 - \cos(x))$$

$$\Rightarrow \cos^2(x) = 1 - \cos^2(x)$$

$$\Rightarrow 2\cos^2(x) = 1$$

$$\Rightarrow \cos^2(x) = \frac{1}{2}$$

$$\Rightarrow \cos(x) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x = 45^\circ \text{ and } 135^\circ$$

- 8 With reference to graph transformations of the function $y = \sin(x)$ explain why the equation $\sin(2x) + \frac{1}{2} = \frac{1}{2}$ has 4 solutions in the interval $0 \leq x \leq 2\pi$.

We want to solve $\sin(2x) = 1 - \frac{1}{2}$ [2 marks]

$\sin(2x)$ is a stretch of $\sin(x)$ by a scale factor $\frac{1}{2}$ parallel to the x -axis.

This equation therefore has 4 solutions in the interval $0 \leq x \leq 2\pi$

- 9 Assuming θ is small enough so that terms involving θ^3 or higher may be ignored, find an expression in terms of θ for $\frac{\sin(2\theta)}{\cos(3\theta)} + \tan(\theta)$. [4 marks]

$$\frac{\sin(2\theta)}{\cos(3\theta)} + \tan(\theta) \approx \frac{2\theta}{\left(1 - \frac{1}{2}(3\theta)^2\right)} + \theta$$

$$= \frac{2\theta}{1 - \frac{9}{2}\theta^2} + \theta$$

$$= 2\theta \left(1 - \frac{9}{2}\theta^2\right)^{-1} + \theta$$

$$= 2\theta \left(1 + \frac{9}{2}\theta^2 + \frac{81}{4}\theta^4\right) + \theta$$

$$\approx 2\theta + 9\theta^3 + \theta$$

$= 3\theta$ ignoring powers of 3 and higher

- 10 Consider the curve given by

$$\sin(x) + \cos(y) = 1, \quad -\pi < x < \pi \text{ and } -2\pi < y < 2\pi.$$

Find the coordinates of the stationary points.

[8 marks]

Differentiate with respect to x

$$\cos(x) - \sin(y) \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos(x)}{\sin(y)}$$

At a stationary point $\frac{dy}{dx} = 0$, so

$$\frac{\cos(x)}{\sin(y)} = 0$$

$$\Rightarrow \cos(x) = 0$$

$$\Rightarrow x = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

When $x = \frac{\pi}{2}$, $\sin\left(\frac{\pi}{2}\right) + \cos(y) = 1$

$$\Rightarrow \cos(y) = 0$$

$$\Rightarrow y = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2} \text{ or } -\frac{3\pi}{2} \text{ in the interval}$$

When $x = -\frac{\pi}{2}$, $\sin\left(-\frac{\pi}{2}\right) + \cos(y) = 1$

$$\Rightarrow \cos(y) = 2 \text{ which has no solutions.}$$

Hence, stationary points are $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$
and $\left(\frac{\pi}{2}, -\frac{3\pi}{2}\right)$

- 11 a) Find where the tangent to the cubic $y = (x+1)(x-2)^2$ at $x = \frac{1}{2}$ meets the x -axis.

[6 marks]

$$\begin{aligned}y &= (x+1)(x-2)^2 \\&= (x+1)(x^2 - 4x + 4) \\&= x^3 - 4x^2 + 4x + x^2 - 4x + 4 \\&= x^3 - 3x^2 + 4\end{aligned}$$

$$\frac{dy}{dx} = 3x^2 - 6x$$

 $\frac{dy}{dx}$

$$\text{When } x = \frac{1}{2}, \left. \frac{dy}{dx} \right|_{x=\frac{1}{2}} = 3 \times \frac{1}{4} - \frac{6}{2} = -\frac{9}{4}$$

Hence, equation of the tangent is of the form,

$y = -\frac{9}{4}x + c$. Passes through $(\frac{1}{2}, \frac{27}{8})$, so

$$\frac{27}{8} = -\frac{9}{4} \times \frac{1}{2} + c$$

$$\Rightarrow c = \frac{27}{8} + \frac{9}{8} = \frac{9}{2}$$

So the equation of the tangent is $y = -\frac{9}{4}x + \frac{9}{2}$.

This tangent meets the x -axis when $y=0$, hence,

$$0 = -\frac{9}{4}x + \frac{9}{2} \Rightarrow \frac{9}{4}x = \frac{9}{2}$$

$$\Rightarrow x = 2$$

So the tangent meets the x -axis at $(2, 0)$.

- b) What is significant about this value of x .

[2 marks]

This value of x is one of the roots of the equation

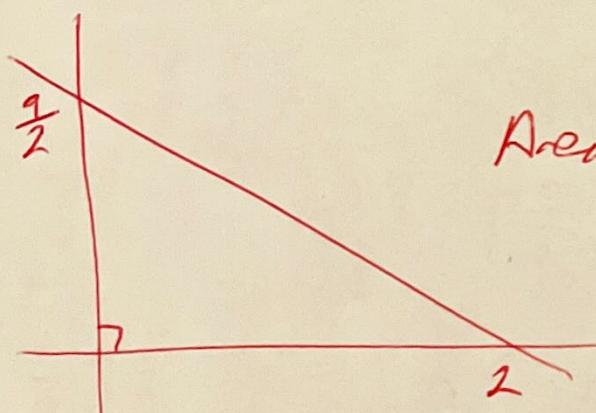
- c) Find the area bound by this tangent and the x - and y -axes.

[4 marks]

The tangent intercepts the y -axis when $x=0$, i.e.

$$y = -\frac{9}{4}x + \frac{9}{2}$$

$$= \frac{9}{2}$$



$$\text{Area} = \frac{1}{2} \times 2 \times \frac{9}{2}$$

$$= \frac{9}{2} \text{ square units}$$

- 12 a) The gradient function of a curve is given by

$$\frac{dy}{dx} = (1 + 2y)(3x + 1).$$

Show that y can be written in the form $y = Ae^{3x^2+2x} - \frac{1}{2}$
[7 marks]

$$\frac{dy}{dx} = (1 + 2y)(3x + 1)$$

$$\Rightarrow \int \frac{1}{1+2y} \frac{dy}{dx} dx = \int 3x + 1 dx$$

$$\Rightarrow \frac{1}{2} \ln|1+2y| = \frac{3x^2}{2} + x + C$$

$$\Rightarrow \ln|1+2y| = 3x^2 + 2x$$

Taking exponentials to the base, e ,

$$1+2y = e^{3x^2+2x+C}$$

$$\Rightarrow 1+2y = Ae^{3x^2+2x} - 1$$

$$\Rightarrow 2y = Ae^{3x^2+2x} - 1$$

$$\Rightarrow y = Ae^{\frac{3x^2+2x}{2}} - \frac{1}{2}$$

b) Given that $y(1) = 6$, find the value of the constant A .

[2 marks]

$$4 \quad y(1) = 6$$

$$6 = Ae^s - \frac{1}{2}$$

$$\Rightarrow Ae^s = \frac{13}{2}$$

$$\Rightarrow A = \frac{13}{2e^s}$$

- 13 a) Show that $I = \int_1^2 ((x+3)\ln(x) + 2) dx = \ln(256) - \frac{7}{4}$
[5 marks]

$$I = \int_1^2 (x+3)\ln(x) + 2 dx$$

~~Let~~ Let $u = \ln(x)$ $\frac{du}{dx} = x+3$
~~du~~ $\frac{du}{dx} = \frac{1}{x}$ $V = \frac{x^2}{2} + 3x$

then using parts

$$I = \left[\left(\frac{x^2}{2} + 3x \right) \ln(x) \right]_1^2 - \int_1^2 \left(\frac{x^2}{2} + 3x \right) dx + \int_1^2 2 dx$$

$$= \left[8\ln(2) - \frac{7}{2}\ln(1) \right] - \left[\left(\frac{1}{4}x^2 + 3x \right) \right]_1^2 + [2x]_1^2,$$

$$= \left(\ln(2^8) - \frac{7}{2}\ln(1) \right) - \cancel{\left(\cancel{(\cancel{1})} \cancel{(\cancel{1})} - \cancel{(\cancel{2})} \right)} - \left(7 - \frac{13}{4} \right) + (4 - 2)$$

$$= \ln(256) - \frac{7}{4}$$

- b) Find the percentage error in approximating I by using the Trapezium Rule with 5 ordinates.

$$f(x) = (x+3)\ln(x) + 2$$

[6 marks]

| x_i | y_i |
|-------|-------------|
| 1 | 2 |
| 1.25 | 2.948360093 |
| 1.5 | 3.824592986 |
| 1.75 | 4.658174993 |
| 2 | 5.465735903 |

So, by the trapezium rule

$$\text{Area} \approx \frac{1}{2} \times \frac{1}{4} \times \left(2 + 5.466 + 2(2.948 + 3.825 + 4.658) \right)$$

$$= 3.790999005$$

 S_0

$$\text{1. error} = \left(\frac{\left(\ln(2.56) - \frac{2}{4} \right) - 3.790999005}{\ln(2.56) - \frac{2}{4}} \right) \times 100$$

≈ 0.111 to 2dp.

- c) What would happen to this percentage error if the number of ordinates used in the Trapezium Rule was increased?

[1 mark]

The percentage error would decrease as the numerical integration would increase in accuracy.

- 14 Using the substitution $u = \tan(x)$ integrate

$$\int \sec^2(x) + 2\tan^2(x)\sec^2(x) + \tan^4(x)\sec^2(x) \, dx$$

[6 marks]

Let $u = \tan(x)$

$$\Rightarrow \frac{du}{dx} = \sec^2(x)$$

$$\Rightarrow dx = \frac{du}{\sec^2(x)}$$

Hence

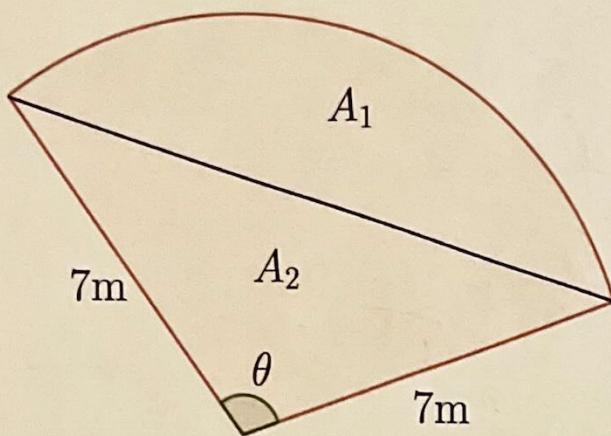
$$I = \int (1 + 2\tan^2(x) + \tan^4(x)) \sec^2(x) \, dx$$

$$= \int (1 + 2u + u^4) \, du$$

$$= u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$$

$$= \tan(x) + \frac{2\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C$$

- 15 The diagram below shows a large flowerbed that is in the shape of the sector of a circle. The gardener wishes to plant Tulips of one colour in the area A_1 and tulips of another colour in the area A_2 .



- a) Show that $98 \sin(\theta) - 49\theta = 0$

[4 marks]

$$A_2 = \frac{1}{2} r^2 \sin(\theta) = \frac{49}{2} \sin(\theta)$$

$$A_1 = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin(\theta) = \frac{49\theta}{2} - \frac{49}{2} \sin(\theta)$$

but due to the planting restrictions $A_1 = A_2$, and so

$$\frac{49\theta}{2} = \frac{49}{2} \sin(\theta) = \frac{49 \sin(\theta)}{2}$$

$$\Rightarrow \frac{49\theta}{2} = 49 \sin(\theta)$$

$$\Rightarrow \theta = 98 \sin(\theta) - 49\theta$$

- b) Show that a root of this equation lies in the interval $[1.5, 2]$.

[2 marks]

$$\text{Let } f(\theta) = 98 \sin(\theta) - 49\theta$$

$$f(1.5) \approx 24.21$$

$$f(2) \approx -8.89$$

As f is continuous and there is a sign change
there is a root $\Rightarrow f(x) = 0$ in the interval $[1.5, 2]$

- c) Find a better approximation to the root using the Newton-Raphson method twice, starting with $x_0 = 1.5$.

[4 marks]

$$f(x) = 98 \sin(x) - 49x$$

$$f'(x) = 98 \cos(x) - 49$$

$$\begin{aligned} \text{So } x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{98 \sin(x_n) - 49x_n}{98 \cos(x_n) - 49} \end{aligned}$$

s_0

$$x_0 = 1.5$$

$$x_1 = 2.076558201$$

$$x_2 = 1.910506616$$