

Solutions

## **AQA A-Level Maths 2022 Paper 1 B**

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

**Name:**

(2, - 6)

**Total Marks:** / 100

What are the coordinates of the minimum point of the function  $y = f(x) + 3$ ?

(3, - 12)

(3, - 9)

(1, - 9)

(3, - 3)

[1 mark]



- 1 Let the sequence  $\{u_n\}$  be defined by the recurrence relation  
 $u_{n+1} = \frac{u_n}{u_{n-1}}$  with  $u_1 = 4$  and  $u_2 = 7$ , then the period of  $\{u_n\}$  is:

1

2

4

6

[1 mark]

$$u_1 = 4$$

$$u_2 = 7$$

$$u_3 = 1.75$$

$$u_4 = 0.25$$

$$u_5 = 4$$

$$u_6 = 7$$

$$u_7 = 4$$

- 2 The coordinates of there minimum point of a polynomial  $p(x)$  are  $(2, -6)$ .

What are the coordinates of the minimum point of  $2f(x - 1) + 3$  ?

(3, -12)

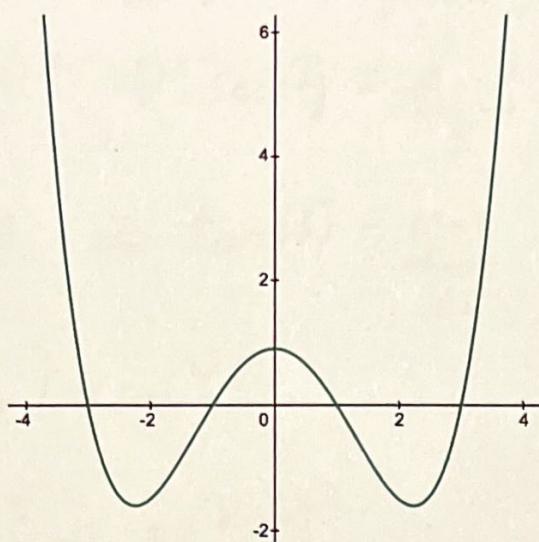
(3, -9)

(1, -9)

(3, -3)

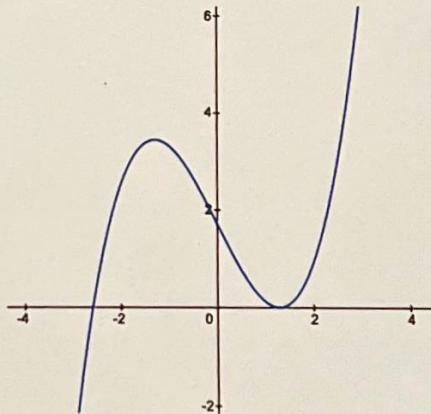
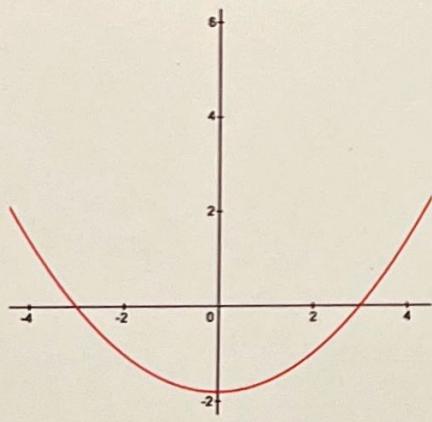
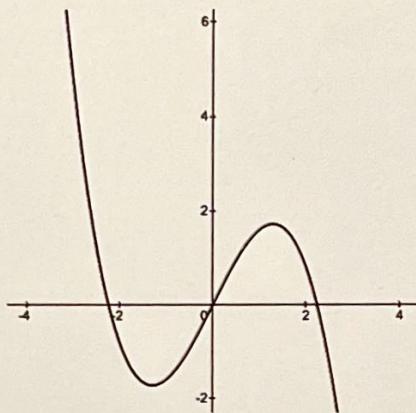
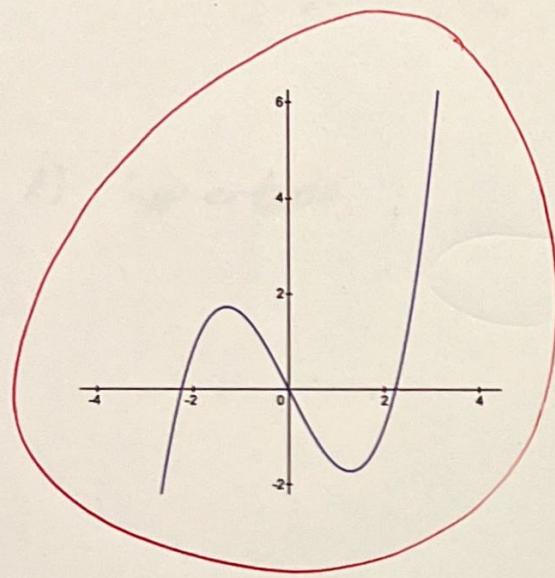
[1 mark]

- 3 The function  $f(x)$  is shown below



Which of the following is the gradient function of  $f(x)$ ?

(1 mark)



- 4 Find the Cartesian equation of the curve with parametric equations  $x = 2 + 3 \sec(t)$ ,  $y = 1 + 4 \tan(t)$ ,  $0 \leq t \leq 2\pi$ .

[4 marks]

$$x = 2 + 3 \sec(t) \Rightarrow \sec(t) = \frac{x-2}{3}$$

$$y = 1 + 4 \tan(t) \Rightarrow \tan(t) = \frac{y-1}{4}$$

$$\text{But } \tan^2(t) + 1 = \sec^2(t)$$

$$\Rightarrow \left(\frac{y-1}{4}\right)^2 + 1 = \left(\frac{x-2}{3}\right)^2$$

$$\Rightarrow \left(\frac{x-2}{3}\right)^2 - \left(\frac{y-1}{4}\right)^2 = 1$$

D hyperbola

- b) Hence, fully factorise  $p(x)$

[2 marks]

5 Let  $p(x) = 2x^3 + ax^2 + bx + 60$

- a) Given that  $(x - 5)$  is a factor of  $p(x)$  and  $p\left(\frac{3}{2}\right) = 0$ , find the values of  $a$  and  $b$ .

[5 marks]

By the factor theorem if  $(x - 5)$  is a factor then  
 $p(5) = 0$ , hence

$$0 = 250 + 25a + 5b + 60 \Rightarrow 25a + 5b = -310 \quad (1)$$

$$p\left(\frac{3}{2}\right) = 0 \Rightarrow 0 = \frac{27}{4} + \frac{9}{4}a + \frac{3}{2}b + 60 \Rightarrow \frac{9}{4}a + \frac{3}{2}b = -\frac{267}{4} \quad (2)$$

Solving (1) and (2) simultaneously gives

$$a = -5 \text{ and } b = -37$$

Hence,

$$\begin{aligned} p(x) &= 2x^3 + ax^2 + bx + 60 \\ &= 2x^3 - 5x^2 - 37x + 60 \end{aligned}$$

- b) Hence, fully factorise  $p(x)$

[2 marks]

$$2x^3 - 5x^2 - 37x + 60 = (x+4)(2x-3)(x-5)$$

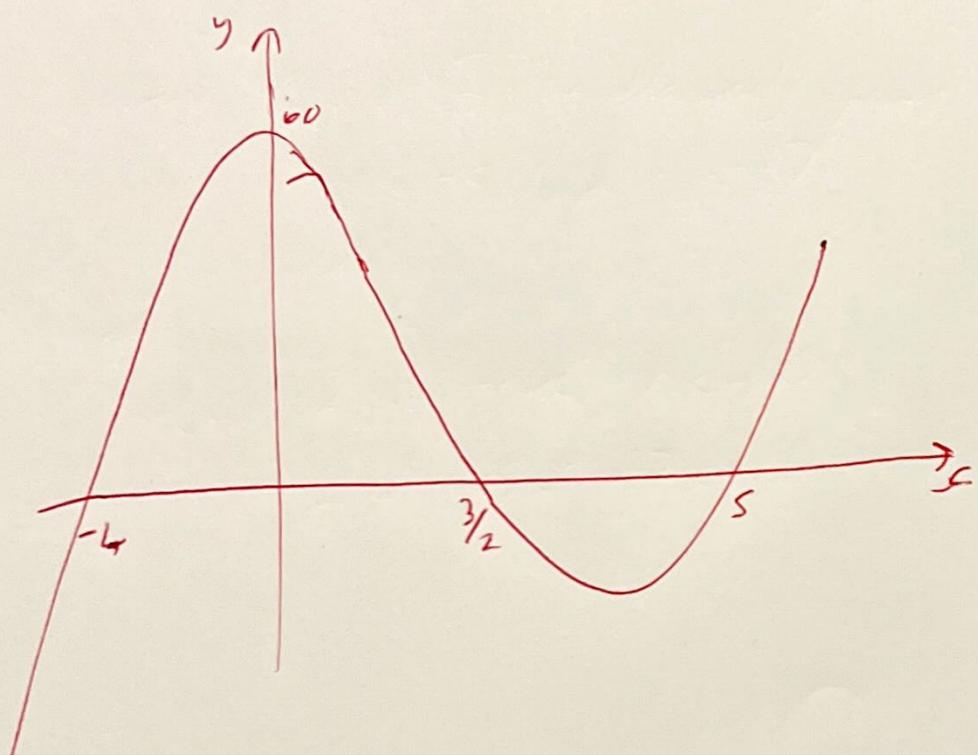
Given that the graph of  $y = p(x)$  is shown below, sketch the graph of  $y = -p(x + 3) + 2$  from the graph of  $y = p(x)$ . [3 marks]

- c) Sketch  $p(x)$ , giving the coordinates of all intersections with the axes.

[3 marks]

roots at  $(-4, 0), \left(\frac{3}{2}, 0\right)$  and  $(5, 0)$

Crosses  $y$ -axis at  $(0, 6)$



- 6 Describe how to obtain the graph of  $y = 3 \sin\left(x - \frac{\pi}{2}\right)$  from the graph of  $y = \sin(x)$ .

[4 marks]

A translation by the vector  $\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$  followed by a stretch scale factor 3 parallel to the y-axis.

- c) Hence, find a polynomial approximation for  $\frac{1}{(2 + \cos x)}^3$  which is valid for small angles. Any terms of a greater order than  $x^4$  may be disregarded.

[3 marks]

- 7 a) Expand  $(3 - x)^{-3}$  using the binomial theorem. Your expansion should show terms up to, and including,  $x^3$ .

[4 marks]

$$(3 - x)^{-3} = \frac{1}{27} + \frac{x}{27} + \frac{2x^2}{81} + \frac{10x^3}{729}$$

- a) After how many years will there be more than £1050 in the account?

[3 marks]

- b) Hence, find a polynomial approximation for  $\frac{1}{(2 + \cos(x))^3}$  which is valid for small angles. Any terms of a greater order than  $x^4$  may be disregarded.

[3 marks]

Using the small angle approximation,

$$2 + \cos(\theta) = 2 + 1 - \frac{\theta^2}{2} = 3 - \frac{\theta^2}{2}$$

Now, using (a)

$$\begin{aligned} \frac{1}{(2 + \cos(\theta))^2} &= (2 + \cos(\theta))^{-3} = \left(3 - \frac{\theta^2}{2}\right)^{-3} \\ &= \frac{1}{27} + \frac{1}{27} \left(\frac{\theta^2}{2}\right) + \frac{2}{81} \left(\frac{\theta^2}{2}\right)^2 \\ &= \frac{1}{27} + \frac{x^2}{54} + \frac{x^4}{162} \end{aligned}$$

- 8 Flora invests £2000 in a savings account which pays simple interest at a rate of 2.5%.

- a) Assuming no withdrawals are made, how much will be in Flora's account after 6 years

[3 marks]

Arithmetic series with  $a = 2050$  [amount after 1 year]  
 $d = 2000 \times 0.025$   
 $= 50$

Then

$$u_6 = 2050 + (6-1) \times 50 \\ = 2300$$

- b) After how many years will there be more than £3060 in the account?

[3 marks]

$$3060 = 2050 + (n-1) \times 50$$

$$\Rightarrow 1010 = (n-1) \times 50$$

$$\Rightarrow n-1 = \frac{1010}{50}$$

$$\Rightarrow n = 1 + \frac{1010}{50}$$

$$= 21.2$$

So after 22 years.

- 9 Show that the curve  $x^2y + 3xy = 10$  has only one stationary point and find its coordinates.

[8 marks]

Differentiating implicitly,

$$x^2 \frac{dy}{dx} + 2xy + 3x \frac{dy}{dx} + 3y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3y - 2xy}{x^2 + 3x}$$

At a stationary point  $\frac{dy}{dx} = 0$ , so

$$-3y - 2xy = 0$$

$$\Rightarrow y(-3 - 2x) = 0$$

$$\Rightarrow y=0 \text{ or } x = -\frac{3}{2}$$

$y=0$  doesn't satisfy  $x^2y + 3xy = 10$  and so can't be the  $x$ -coordinate of a stationary point.

When  $x = -\frac{3}{2}$ ,

$$\frac{9}{4}y - \frac{9}{2}y = 10 \Rightarrow y = -\frac{40}{9}$$

Hence, the stationary point has coordinates  $\left(-\frac{3}{2}, -\frac{40}{9}\right)$

10 Find, by using the substitution  $x = 2 \sin(\theta)$ , the definite integral

$$\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4 - x^2} dx$$

[10 marks]

Let  $x = 2 \sin(\theta) \Rightarrow \frac{dx}{d\theta} = 2 \cos(\theta) \Rightarrow dx = 2 \cos(\theta) d\theta$

For the limits : Upper  $\theta = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Lower  $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right) = \frac{-\pi}{3}$ , hence

$$\begin{aligned} I &= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4 - x^2} dx \\ &= 2 \int_{-\pi/3}^{\pi/3} \sqrt{4 - 4 \sin^2 \theta} \cos(\theta) d\theta \\ &= 2 \int_{-\pi/3}^{\pi/3} 2 \cos^2 \theta d\theta \\ &= 4 \int_{-\pi/3}^{\pi/3} \frac{1}{2} \cos(2\theta) + \frac{1}{2} d\theta \\ &= \left[ \sin(2\theta) + 2\theta \right]_{-\pi/3}^{\pi/3} \\ &= \left( \frac{\sqrt{3}}{2} + \frac{2\pi}{3} \right) - \left( -\frac{\sqrt{3}}{2} - \frac{2\pi}{3} \right) \\ &= \sqrt{3} + \frac{4\pi}{3} \end{aligned}$$



11

a) Find the integral  $\int_{-3}^1 (x+2)(x-1)(x+3) dx$

[2 marks]

$$\begin{aligned} \int_{-3}^1 (x+2)(x-1)(x+3) dx &= \int_{-3}^1 x^3 + 6x^2 + x - 6 dx \\ &= \left[ \frac{x^4}{4} + \frac{6x^3}{3} - \frac{x^2}{2} - 6x \right]_{-3}^1 \\ &= -\frac{32}{3} \end{aligned}$$

b) Approximate this integral using the trapezium rule with 4 strips.

[3 marks]

$$h = \frac{-3-1}{4} = 1$$

$x$	$f(x)$
-3	0
-2	0
-1	-4
0	-6
1	0

Using the trapezium rule we have

$$A \approx \frac{1}{2} [0 + 0 + 2(0 + 4 + 6)] \\ = 10$$

- c) Calculate the percentage error made in the approximation by the trapezium rule with 4 strips.

[1 marks]

$$\% \text{ error} = \frac{\left| -\frac{32}{3} - 10 \right|}{\left| -\frac{32}{3} \right|} \times 100$$

$$= 6.25\%$$

- d) What would the error be if using the trapezium rule for the integral of a linear function. Explain your answer.

[2 marks]

The area would be zero since the trapezium rule is exact for linear functions.

- e) Explain why, with reference to a diagram, the integral computed in (a) is not the area between the curve

$$y = (x+3)(x-1)(x+2)$$

[2 marks]



The area between the curve and the x-axis consists of areas below and above the axis. So the integral needs to be split into two and the absolute value of the integral for the regions below the axis needs to be considered.

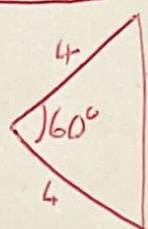
- 12 A sequence of triangular tiles is to be cut from a sheet of ABS plastic.

The first tile to be cut has two side lengths of 4 cm with an angle of  $60^\circ$  between these ~~sides~~<sup>edges</sup>. Subsequent tiles are made by enlarging the previous tile by a scale factor of a  $\frac{1}{2}$ .

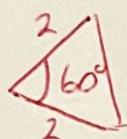
Show that all tiles can be cut from a sheet of ABS of area  $10 \text{ cm}^2$ .

[6 marks]

Tile 1:



Tile 2



Tile 3



$$\text{Area} = 4\sqrt{3}$$

$$\text{Area} = \sqrt{3}$$

$$\text{Area} = \frac{\sqrt{3}}{4}$$

Geometric series

$$a = 4\sqrt{3}$$

$$r = \frac{1}{4}$$

Converge

$$S_{\infty} = \frac{a}{1-r}$$

$$= \frac{4\sqrt{3}}{\frac{3}{4}}$$

$$= \frac{16\sqrt{3}}{3}$$

$$< 10$$

Of course this ignores any practical considerations of arranging tiles within  $10 \text{ cm}^2$  or the ability to cut increasingly smaller tiles.

15) The points A(2, 1) and B(6, 2) lie on the circumference of a circle.

- i) Find the equation of the perpendicular bisector of A and B.  
[4 marks]

- ii) Given that the line  $-5x + 9y = -17$  passes through the centre of the circle, find the centre and radius of the circle. Fully justify your answers.

[4 marks]

- 13 The points  $A(2,6)$  and  $B(6,2)$  lie on the circumference of a circle.

- a) Find the equation of the perpendicular bisector of  $A$  and  $B$   
**[4 marks]**

$$\text{Gradient of } AB = \frac{2-6}{6-2} = -1$$

So gradient of  $\perp$  bisector is  $m=1$

$$\text{Midpoint of } AB = \left(\frac{2+6}{2}, \frac{6+2}{2}\right) = (4, 4)$$

Hence eq'n of bisector of the form

$$y = x + c$$

passing through  $(4, 4)$ , hence  $c < 0$  and  
equation of bisector is  $y = x$

- b) Given that the line  $-5x + 9y = -12$  passes through the centre of the circle, find the centre and radius of the circle. Fully justify your answer.

**[4 marks]**

Since  $A$  and  $B$  lie on the circumference of the circle  $AB$  is a chord and so the perpendicular bisector of  $AB$  passes through the centre

Since  $-5x + 9y = -12$  also passes through the centre

$$\begin{aligned} -5x + 9x &= -12 \\ \Rightarrow 4x &= -12 \\ x &= -3 \end{aligned}$$

and  $y = -3$

so  $(-3, -3)$  is the intersection of the lines  $y = x$  and  $-5x + 9y = -12$ , and is therefore the centre of the circle.

$$\begin{aligned} \text{Radius} &= \sqrt{(6 - -3)^2 + (2 - -3)^2} \\ &= \sqrt{9^2 + 5^2} \\ &= \sqrt{106} \end{aligned}$$

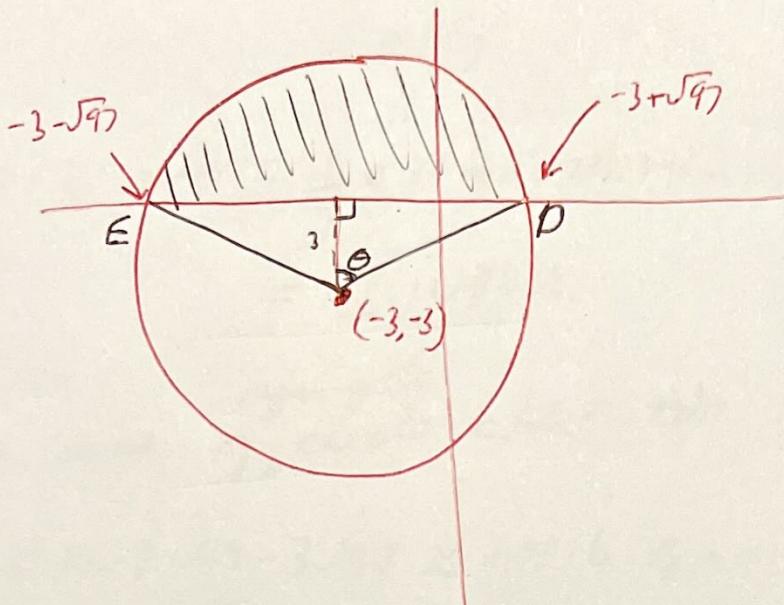
- c) Hence, write down the equation of the circle.

**[1 mark]**

$$(x + 3)^2 + (y - 3)^2 = 106$$

- d) Find the area of the portion of the circle which is above the  $x$ -axis.

[6 marks]



Circle crosses  $x$ -axis at  $E$  and  $D$ . At these points,  $y=0$ , hence

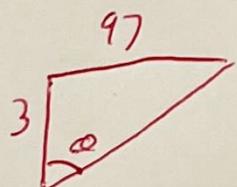
$$(x+3)^2 + 3^2 = 10^2$$

$$\Rightarrow (x+3)^2 = 97$$

$$\Rightarrow x+3 = \pm\sqrt{97}$$

$$\Rightarrow x = -3 \pm \sqrt{97}$$

To find  $\theta$  consider



$$\therefore \theta = \arctan\left(\frac{\sqrt{97}}{3}\right)$$

$$= 1.275121187^\circ$$

$$\text{Area of triangle} = \frac{1}{2} \times 3 \times 2\sqrt{97}$$

$$= 3\sqrt{97}$$

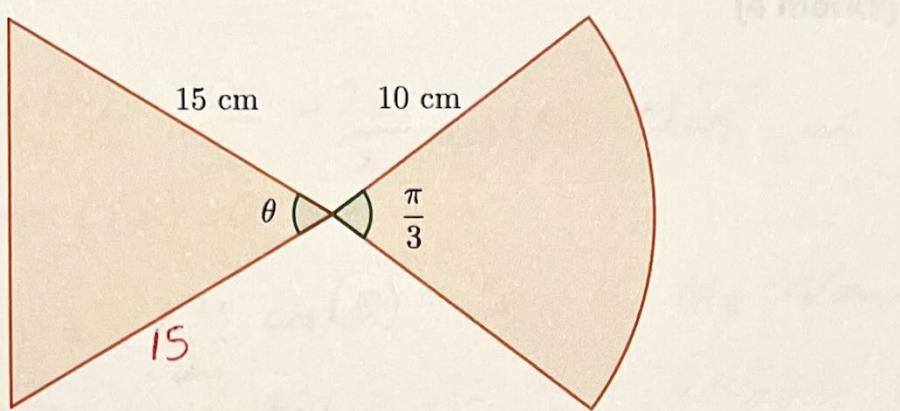
$$\text{Area of sector} = \frac{1}{2} \times 106 \times 1.27512118$$

$$= 135.1628459$$

So the area of the <sup>region of the</sup> circle above the x-axis is

$$135.1628459 - 3\sqrt{97} \approx 105.6 \text{ square units}$$

- 14 A craftsperson wants to make the design shown below. It is required that the triangle is the same area as the sector of the circle.



a) Show that  $\frac{50\pi}{3} = \frac{225}{2} \sin(\theta)$

[3 marks]

$$\text{Area of triangle} = \frac{1}{2} \times 15^2 \times \sin(\theta)$$

$$= \frac{1}{2} \times 15^2 \times \sin(\theta)$$

$$= \frac{225}{2} \sin(\theta)$$

$$\text{Area of sector} = \frac{1}{2} \times 10^2 \times \frac{\pi}{3}$$

$$= \frac{50\pi}{3}$$

As areas are equal, we have

$$\frac{50\pi}{3} = \frac{225}{2} \sin(\theta)$$

- b) Set up a Newton-Raphson method to solve the equation derived in part (a). Use  $x_1 = \frac{\pi}{3}$  and complete 3 iterations.

[4 marks]

Let  $f(x) = \frac{50\pi}{3} - \frac{225}{2} \sin(x)$ , then we are solving  $f(x) = 0$ .

$f'(x) = -\frac{225}{2} \cos(x)$ . So, by the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{\left( \frac{50\pi}{3} - \frac{225}{2} \sin(x_n) \right)}{-\frac{225}{2} \cos(x_n)}$$

With  $x_1 = \frac{\pi}{3}$ , we have

$$x_1 = \frac{\pi}{3}$$

$$x_2 = 0.2459890114$$

$$x_3 = 0.4747818216$$

$$x_4 = 0.4840878616$$

- c) Solve the equation exactly and determine to how many decimal places the Newton-Raphson solution is correct to?

[2 marks]

$$\frac{50\pi}{3} = \frac{225}{2} \sin(\theta)$$

$$\Rightarrow \sin(\theta) = \frac{100\pi}{3 \times 225}$$

$$\Rightarrow \theta = \arcsin\left(\frac{100\pi}{3 \times 225}\right)$$

$$= 0.4841103612$$

So our solution is accurate to 3dp.

- 15 The tangent to the curve  $y = (x+2)(x-3)(x-6)$  at the point  $A$ , where  $x = 4$  meets the curve again at the point  $B$ . Find the distance  $|AB|$ .

[8 marks]

$$\begin{aligned}y &= (x+2)(x-3)(x-6) \\&= x^3 - 7x^2 + 36\end{aligned}$$

$$\text{When } x = 4, \quad y = -12$$

$$\frac{dy}{dx} = 3x^2 - 14x$$

and so

$$\left. \frac{dy}{dx} \right|_{x=4} = -8$$

Equation of tangent is of the form

$$y = -8x + C$$

passes through  $(4, -12)$  so  $C = 20$

Equation of tangent is therefore

$$y = -8x + 20$$

Substitute this into the equation of the curve

$$-8x + 20 = x^3 - 7x^2 + 36$$

$$\Rightarrow x^3 - 7x^2 + 8x + 16 = 0$$

$$\Rightarrow (x-4)^2(x+1) = 0$$

so the tangent meets the curve again when  
 $s = -1$ .

When  $x = -1$ ,  $y = -14$ .

so the tangent at the point where  $s = 4$  meets the curve again at  $(-1, -14)$